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Number Theory I - Solutions

1 Divisibility

Beginner

1.1 Show that 900 divides 10!.

Solution:

$$900 = 2 \cdot 5 \cdot 9 \cdot 10 \mid 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 10!$$

1.2 The product of two numbers, neither of which is divisible by 10, is 1000. Determine the sum of these numbers.

Solution: Let's call our two numbers a and b. The prime factorisation of 1000 is $1000 = 2^3 5^3$. It follows that:

$$a = 2^k 5^l \text{ and } b = 2^{3-k} 5^{3-l}$$

for natural numbers $k, l \in \{0, 1, 2, 3\}$. Now if k and l strictly greater than 0, then a would be divisible by 10, which is a contradiction. Otherwise, if k and l are strictly smaller than 3, then b is divisible by 10, which is also a contradiction. Thus we get than (k, l) is either $(3, 0)$ or $(0, 3)$. In the first case, we have $a = 8$ and $b = 125$. In the second, $a = 125$ and $b = 8$. In both cases, we get that $a + b = 133$.

1.3 Find all natural numbers n, such that n is a divisor of $n^2 + 3n + 27$.

Solution: Since n divides the first two terms of $n^2 + 3n + 27$, we have that:

$$n \mid n^2 + 3n + 27 \Leftrightarrow n \mid 27.$$

Since $27 = 3^3$, this second statement is valid if and only if $n \in \{1, 3, 9, 27\}$. Therefore n is a divisor of $n^2 + 3n + 27$ if and only if $n \in \{1, 3, 9, 27\}$.

Advanced

1.4 Show that:

(a) $5 \cdot 17 \mid 5^2 \cdot 17 + 3 \cdot 5 \cdot 9 + 5 \cdot 3 \cdot 8$

(b) $n(n + m) \mid 3mn^2 + amn^2 + 3n^3 + an^3$

Solution:

(a) We can rewrite the right-hand side as:

$$5^2 \cdot 17 + 3 \cdot 5 \cdot 9 + 5 \cdot 3 \cdot 8 = 5^2 \cdot 17 + 5 \cdot 3(9 + 8) = 5^2 \cdot 17 + 5 \cdot 3 \cdot 17 = 5 \cdot 17(5 + 3)$$

which is visibly divisible by $5 \cdot 17$.

(b) Again we rewrite the right-hand side:

$$\begin{aligned}3mn^2 + amn^2 + 3n^3 + an^3 &= n^2(3m + am + 3n + an)\\ &= n^2\left((3+a)m + (3+a)n\right)\\ &= (3+a)n^2(m+n)\\ &= n(n+m)\left((3+a)n\right)\end{aligned}$$

which in turn is visibly divisible by $n(n+m)$.

1.5 Find three three-digit natural numbers whose decimal representations uses nine different digits, and such that their product ends with four zeros.

Solution: Let's construct the three numbers a, b, c in such a way that $10 \mid a$, $2^3 \mid b$ and $5^3 \mid c$. This give us what we want:

$$10000 = 10 \cdot 2^3 \cdot 5^3 \mid abc.$$

Let's first set $c = 125$ and look for a, multiple of 10, and b, multiple of 8, such that all the digits are different. One solution for example would be $b = 864 = 108 \cdot 8$ and $a = 370 = 37 \cdot 10$. This ultimately gives us the solution $(370, 864, 125)$.

1.6 (a) Find all natural numbers who have exactly 41 divisors and that are divisible by 41.

(b) Find all natural numbers who have exactly 42 divisors and that are divisible by 42.

Solution:

(a) Consider the unique prime factorisation of our number:

$$41^n \cdot p_1^{\alpha_1} \cdot \dots \cdot p_k^{\alpha_k},$$

with $p_1, \dots, p_k$ primes different from 41. Since the number must be divisible by 41, we have that $n \geq 1$. Furthermore, our number has exactly $(n+1)(\alpha_1+1)\dots(\alpha_k+1)$ distinct divisors. This gives us the equation:

$$41 = (n+1)(\alpha_1+1)\dots(\alpha_k+1).$$

Since 41 is prime, and $(n+1) \geq 2$, we get that $(n+1) = 41$ and $\alpha_1 = \dots = \alpha_k = 0$. Thus, 41^{40} is the only number that satisfies our conditions.

(b) We note that $42 = 2 \cdot 3 \cdot 7$. Like above, our number will contain the following prime factors:

$$2^\alpha \cdot 3^\beta \cdot 7^\gamma$$

where α, β, $\gamma \geq 1$. Since 42 can only be factored as $42 = 2 \cdot 3 \cdot 7$, we can't have any other prime factors, and our solutions are therefore:

$$2 \cdot 3^2 \cdot 7^6,\ 2 \cdot 3^6 \cdot 7^2,\ 2^2 \cdot 3 \cdot 7^6,\ 2^2 \cdot 3^6 \cdot 7,\ 2^6 \cdot 3 \cdot 7^2,\ 2^6 \cdot 3^2 \cdot 7.$$

Olympiad

1.7 Find all natural numbers n such that $n+1 \mid n^2+1$.

Solution: We know that $n+1 \mid n^2+1$ et we can assert that $n+1 \mid n(n+1) = n^2+n$. Thus $n+1$ also divides the difference:

$$n+1 \mid \left(n^2+n\right) - \left(n^2+1\right) = n-1.$$

For $n > 1$, we get that $n+1 > n-1 > 0$, in which case $n+1$ cannot be a divisor of $n-1$. Thus we only need to consider $n = 1$, and indeed $2 \mid 2$.

1.8 Show that for all natural numbers n, there are n consecutive natural numbers such that none of them are prime.

Solution: Consider the following set of n consecutive natural numbers:

$$M = \{(n+1)! + 2\ ,\ (n+1)! + 3\ ,\ \dots\ ,\ (n+1)! + (n+1)\}.$$

By definition, $(n+1)!$ is divisible by all of $\{1, 2, \dots, (n+1)\}$. None of the numbers in M is prime as for all $k \in \{2, 3, \dots, (n+1)\}$, the numbers $(n+1)! + k$ is divisible by k as k divides both $(n+1)!$ and k.

1.9 Show that there are infinitely many natural numbers n, such that $2n$ is a square number, $3n$ a cube, and $5n$ a fifth power.

Solution: Let's start with something like this $n = 2^\alpha \cdot 3^\beta \cdot 5^\gamma$. The three conditions in the exercise give us the following divisibility conditions for α, β and γ:

$$\begin{array}{lll} 2 \mid \alpha + 1 & 3 \mid \alpha & 5 \mid \alpha \\ 2 \mid \beta & 3 \mid \beta + 1 & 5 \mid \beta \\ 2 \mid \gamma & 3 \mid \gamma & 5 \mid \gamma + 1 \end{array}$$

We can see that these conditions are satisfied for example when $\alpha = 15$, $\beta = 20$ and $\gamma = 24$. Thus $n = 2^{15} \cdot 3^{20} \cdot 5^{24}$ is a solution. Then we notice that for a prime p other than 2, 3 or 5, the number $p^{30} \cdot 2^{15} \cdot 3^{20} \cdot 5^{24}$ also satisfies our conditions. And since there are an infinite amount of primes, we get an infinite amount of solutions.

2 gcd and lcm

Beginner

2.1 (IMO 59) Show that for all natural numbers n, the following fraction is irreducible:

$$\frac{21n+4}{14n+3}$$

Solution: We use Euclid's algorithm to show that the gcd of the two numbers is 1, and therefore the two numbers are coprime:

$$\begin{aligned} (21n+4, 14n+3) &= ((21n+4) - (14n+3), 14n+3) \\ &= (7n+1, 14n+3) \\ &= (7n+1, (14n+3) - 2 \cdot (7n+1)) \\ &= (7n+1, 1) = 1. \end{aligned}$$

2.2 Find all pairs of natural numbers (a, b) such that:

$$\operatorname{lcm}(a, b) = 10 \gcd(a, b)$$

Solution: Let $d = \gcd(a, b)$ and let's write $a = dm$, $b = dn$ where $\gcd(m, n) = 1$. Then we have that $\operatorname{lcm}(a, b) = dmn$ and therefore $mn = 10$. That means the solutions are all of the form

$(a, b) = (dm, dn)$, where d, m, n are natural numbers, $\gcd(m, n) = 1$ and $mn = 10$. With that, for all natural numbers d, we get the following solutions:

$$(d, 10d)\ ,\ (2d, 5d)\ ,\ (5d, 2d)\ ,\ (10d, d).$$

Advanced

2.3 Show that every natural number $n > 6$ is the sum of two coprime natural numbers greater than one.

Solution: When n is odd, let's write $n = 2k + 1$ for a natural number $k \geq 3$. Then we have that $n = (k + 1) + k$ where $\gcd(k, k + 1) = 1$ and we're done. When n is even, we have that $n = 2k$ for a natural number $k > 3$. In which case, we split again into two an even and an odd case, this time for k. If k is even, then $k - 1$ is odd and we can write $n = (k - 1) + (k + 1)$. Indeed, we have that $\gcd(k - 1, k + 1) = \gcd(k - 1, 2) = 1$. Furthermore, since $k > 3$, we get $k - 1 > 2$ and we have our solution. If k is odd, then $k - 2$ is also odd and with our assumption, we also have $k - 2 > 1$. Thus the decomposition $n = (k - 2) + (k + 2)$ satisfies the condition: $\gcd(k - 2, k + 2) = \gcd(k - 2, 4) = 1$.

2.4 Two natural numbers a and b are said to be *friends* if $a \cdot b$ is a square number. Show that if a and b are friends, then so are a and $\gcd(a, b)$.

Solution: Let $d = \gcd(a, b)$ and let's write $a = dm$, $b = dn$ where $\gcd(m, n) = 1$. Lettes suppose that $a \cdot b = d^2 \cdot m \cdot n$ is a perfect square. As m and n are coprime, this means that m and n are also perfect squares. Then $a \cdot \gcd(a, b) = d^2 \cdot m$ is also a perfect square, and thus a and $\gcd(a, b)$ are friends.

Olympiad

2.5 Let m and n be two natural numbers whose sum is a prime number. Show that m and n are coprime.

Solution: Suppose that m and n aren't coprime and let $d = \gcd(m, n) \geq 2$. Then we have that $d \mid m$, $d \mid n$, and $d \mid m + n$. Furthermore, $d \leq m$ and $d \leq n$, giving us $d < m + n$. Therefore d is a non-trivial divisors of $m + n$, which means that $m + n$ isn't prime.

2.6 (Canada 97) Find all pairs of natural numbers (x, y) where $x \leq y$ and such that they satisfy the following equations:

$$\gcd(x, y) = 5! \text{ and } \operatorname{lcm}(x, y) = 50!$$

Solution: Let $d = \gcd(x, y)$ and let $x = dm$, $y = dn$ where $\gcd(m, n) = 1$. Then we have that $d = 5!$ and $\operatorname{lcm}(x, y) = dmn = 50!$, so $mn = \frac{50!}{5!} = 6 \cdot 7 \cdot \cdots \cdot 50$. Within the number $\frac{50!}{5!}$, the 15 prime factors in $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$ each appear at least once. Now let p be one of those primes. If m is divisible by p, then by assumption n cannot be divisible by p. It follows that the set P can be partitioned into two parts, which correspond to the chosen factors of m and n. This can be done in 2^{15} different ways. Up until now, we haven't taken into account the fact that $x \leq x$ and therefore $m \leq n$. Clearly $m = n$ is impossible since both numbers have different prime factors. This means that we've counted the number of pairs twice, since when $x > y$, we've counted both (x, y) and (y, x). Therefore, there are 2^{14} different pairs in total that verify the condition.

3 Estimations

Beginner

3.1 A rectangle is said to be *beautiful* if the lengths of all of its sides are natural numbers, and if the measures of its perimeter and area are equal. Find all the *beautiful* rectangles.

Solution: Let a, b be the sides of the rectangle. In a beautiful rectangle, they satisfy the equation $ab = 2(a + b) \Leftrightarrow ab - 2(a + b) = 0$. By symmetry, we can assume WLOG (without loss of generality) that $a \geq b$. Suppose we have that $b \geq 5$. Then we'd get:

$$ab - 2(a + b) = b(a - 2) - 2a \geq 5(a - 2) - 2a = 3a - 10 \geq 15 - 10 > 0$$

And so there is no beautiful rectangle with such sides. So we see that the smallest side of a beautiful rectangle has side length less than 5. Testing all the cases for b $b \in \{1, 2, 3, 4\}$, we get two solutions $(a, b) = (6, 3)$ and $(a, b) = (4, 4)$. By symmetry, $(a, b) = (3, 6)$ is also a solution.

3.2 Find all pairs of natural numbers (x, y) such that:

$$\frac{1}{x} + \frac{2}{y} = 1.$$

Solution: Suppose that $y \geq 3$ and $x \geq 4$. Then the left-hand side is strictly smaller than 1, which is impossible. Thus, either $y \leq 2$ or $x \leq 3$. Testing all the cases, we find that the only solutions are $(x, y) = (2, 4)$ and $(x, y) = (3, 3)$.

Advanced

3.3 A rectangular parallelepiped is said to be *beautiful* if the lengths of all of its sides are natural numbers, and if the measures of its volumes and surface area are equal.Find all the *beautiful* rectangular parallelepipeds.

Solution: Let a, b, c be the sides of a beautiful parallelepiped. This means that they satisfy:

$$abc = 2(ab + bc + ca) \Leftrightarrow abc - 2(ab + bc + ca) = 0.$$

By symmetry we can assume WLOG that $a \leq b \leq c$. Let's also suppose that $a \geq 7$. Then we have:

$$\begin{aligned} abc - 2(ab + bc + ca) &= a(bc - 2b - 2c) - 2bc \geq 7(bc - 2b - 2c) - 2bc) \\ &= 5bc - 14b - 14c = b(5c - 14) - 14c \\ &\geq 35c - 98 - 14c = 21c - 98 \\ &\geq 147 - 98 > 0. \end{aligned}$$

This is a contradiction, and so $a \leq 6$. Now we test the different cases of $a \in \{1, 2, 3, 4, 5, 6\}$:

$a = 1 \Rightarrow bc - 2b - 2c - 2bc = 0 \Rightarrow -(bc + 2b + 2c) = 0$. This is a contradiction because the left-hand side is always negative.

$a = 2 \Rightarrow 2bc - 2b - 2c - 2bc = 0 \Rightarrow -2(b + c) = 0$. Here there left-hand side is also always negative, so no solution.

$a = 3 \Rightarrow 3bc - 6b - 6c - 2bc = 0 \Rightarrow bc - 6b - 6c = 0$. Suppose that $c \geq b \geq 13$. Then we have $bc - 6b - 6c = b(c - 6) - 6c \geq 7c - 78 > 0$, which is a contradiction. What's left to test are the cases $b \in \{3, 4, \dots, 12\}$ which gives us the following solutions for (a, b, c):

$$(3, 7, 42)\ ,\ (3, 8, 24)\ ,\ (3, 9, 18)\ ,\ (3, 10, 15)\ ,\ (3, 12, 12).$$

$a = 4 \Rightarrow 4bc - 8b - 8c - 2bc = 0 \Rightarrow bc - 4b - 4c = 0$. Suppose that $c \geq b \geq 9$. Then we have $bc - 4b - 4c = b(c - 4) - 4c \geq 5c - 36 > 0$, which is a contradiction. What's left to test are the cases $b \in \{4, 5, \dots, 8\}$ which gives us the following solutions for (a, b, c):

$$(4, 5, 20)\ ,\ (4, 6, 12)\ ,\ (4, 8, 8).$$

$a = 5 \Rightarrow 5bc - 10b - 10c - 2bc = 0 \Rightarrow 3bc - 10b - 10c = 0$. Suppose that $c \geq b \geq 7$. Then we have $3bc - 10b - 10c \geq 21c - 70 - 10c = 11c - 70 > 0$, which is a contradiction. What's left to test are the cases $b = 5$ and $b = 6$ which gives us the following solution for (a, b, c):

$$(5, 5, 10).$$

$a = 6 \Rightarrow 6bc - 12b - 12c - 2bc = 0 \Rightarrow bc - 3b - 3c = 0$. Suppose that $c \geq b \geq 7$. Then we have $bc - 3b - 3c = b(c - 3) - 3c \geq 4c - 21 > 0$, which is a contradiction. What's left to test is the case $b = 6$ which gives us the following solution for (a, b, c):

$$(6, 6, 6).$$

Bringing everything together, these are all the solutions for the triplet (a, b, c):

$$(3, 7, 42)\ ,\ (3, 8, 24)\ ,\ (3, 9, 18)\ ,\ (3, 10, 15)\ ,\ (3, 12, 12),$$

$$(4, 5, 20)\ ,\ (4, 6, 12)\ ,\ (4, 8, 8)\ ,\ (5, 5, 10)\ ,\ (6, 6, 6),$$

and all their permutations.

3.4 Find all triplets of natural numbers (x, y, z) such that:

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1.$$

Solution: The equation is equivalent to:

$$\frac{1}{x} + \frac{2}{y} = 1 + \frac{3}{z}.$$

The right-hand side is always strictly greater than 1, and so must the left-hand side. However, if $x \geq 2$ and $y \geq 4$, the left-hand side is at most, which is a contradiction. Thus, either $x = 1$ or $y \leq 3$, and so we test the different cases:

- $x = 1$: We get the equation $\frac{2}{y} = \frac{3}{z} \Leftrightarrow 3y = 2$. In which case, the solutions are $(x, y, z) = (1, 2k, 3k)$ for a natural number k.
- $y = 1$: We get the equation $\frac{1}{x} + 1 = \frac{3}{z}$. The left-hand side is always strictly greater than 1, and so must the right-hand side, thus $z < 3$. This gives us the solution $(x, y, z) = (2, 1, 2)$.
- $y = 2$: We get the equation $\frac{1}{x} = \frac{3}{z} \Leftrightarrow 3x = z$. In which case, the solutions are $(x, y, z) = (k, 2, 3k)$ for a natural number k.

- $y = 3$: We get the equation $\frac{1}{x} = \frac{1}{3} + \frac{3}{z}$. The left-hand side is always strictly greater than $\frac{1}{3}$, and so must the right-hand side, thus $x \leq 2$. This gives us the solution $(x, y, z) = (2, 3, 18)$.

All in all, we get two families of solutions:

$$(x, y, z) = (1, 2k, 3k) \text{ and } (x, y, z) = (k, 2, 3k) \text{ pour } k \geq 1,$$

as well as two particular solutions:

$$(x, y, z) = (2, 1, 2) \text{ and } (x, y, z) = (2, 3, 18).$$

3.5 Find all natural numbers n such that $n^2 + 1$ is a divisor of $n^7 + 13$.

Solution: We can rewrite $n^7 + 13 = (n^2 + 1)(n^5 - n^3 + n) - n + 13$. $n^2 + 1$ is clearly a divisor of $(n^2 + 1)(n^5 - n^3 + n)$. For $n^2 + 1$ to be a divisor of $n^7 + 13$, it is necessary that $(n^2 + 1) \mid -n + 13$. For this to happen, we must either have that $(n^2 + 1) \leq -n + 13 \Leftrightarrow n \leq 3$ or that $-n + 13 = 0 \Leftrightarrow n = 13$. We now test for the cases $n \in \{1, 2, 3, 13\}$ and we see that $n^2 + 1$ is a divisor of $n^7 + 13$ if and only if $n \in \{1, 3, 13\}$.

Olympiad

3.6 Show that the equation

$$y^2 = x(x + 1)(x + 2)(x + 3)$$

has no solution in the natural numbers.

Solution: The right-hand side is equal to $x(x + 1)(x + 2)(x + 3) = x^4 + 6x^3 + 11x^2 + 6x$. Furthermore, we have that

$$(x^2 + 3x)^2 = x^4 + 6x^3 + 9x^2,$$
$$(x^2 + 3x + 1)^2 = x^4 + 6x^3 + 11x^2 + 6x + 1.$$

Thus:

$$(x^2 + 3x)^2 < x(x + 1)(x + 2)(x + 3) < (x^2 + 3x + 1)^2$$

For all natural numbers x, which means that $x(x+1)(x+2)(x+3)$ can never be a perfect square.

3.7 Find all integers x for which

$$x! = x^2 + 11x - 36$$

Solution: Suppose that $x \geq 5$. Then we have that:

$$x! - x^2 - 11x + 36 > x(x - 1)(x - 2) - x^2 - 11x + 36 = x^3 - 4x^2 - 9x + 36 > 0,$$

which is a contradiction. Testing all the cases for $x \in \{1, 2, 3, 4\}$, we get that $x = 3$ and $x = 4$.

3.8 (IMO 98) Find all pairs of natural numbers (a, b) such that $a^2b + a + b$ is divisible by $ab^2 + b + 7$.

Solution: We have that:

$$ab^2 + b + 7 \mid b \cdot (a^2b + a + b) - a \cdot (ab^2 - b - 7) = b^2 - 7a.$$

Moreover $ab^2 + b + 7 \geq b^2 - 7a$ for all natural numbers a and b. Thus the condition is only verified when $b^2 - 7a \leq 0$. If $b^2 - 7a = 0$, then we get a family of solutions $(a, b) = (7k^2, 7k)$ where k is a natural number. If $b^2 - 7a < 0$, then we would need $7a - b^2 \geq ab^2 + b + 7$ for the divisibility condition to be satisfied. This is equivalent to:

$$(a+1)b^2 + b + 7 - 7a \leq 0.$$

For $b \geq 3$, we get $(a+1)b^2 + b + 7 - 7a \geq 2a + 17 > 0$, in which case the inequation above cannot be verified. So we only need to test for $b = 2$ and $b = 1$.

For $b = 2$, we get $4a + 9 \,|\, 2a^2 + a + 2$ thanks to the original divisibility condition. This however gives us:

$$4a + 9 \,|\, 2 \cdot (2a^2 + a + 2) - (a - 1) \cdot (4a + 9) = 3a - 5.$$

But this condition is never satisfied as for all natural numbers a, the left-hand side is strictly greater than the right-hand side and the right-hand side is never zero. Thus $b = 2$ gives no solution.

For $b = 1$, we get $a + 8 \,|\, a^2 + a + 1$ and thus:

$$a + 8 \,|\, (a^2 + a + 1) - (a - 7) \cdot (a + 8) = 57.$$

Since $57 = 3 \cdot 19$, we get $a = 11$ or $a = 49$.

All in all, we're left with the family of solutions:

$$(a, b) = (7k^2, 7k) \text{ for } k \in \mathbb{N},$$

as well as two particular solutions:

$$(a, b) = (11, 1) \text{ and } (a, b) = (49, 1).$$