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Number Theory I - Hints

1 Divisibility

Beginner

1.1 Show that 900 divides 10!.

Hint: Write 900 as the product of *distinct* numbers in $\{1, 2, \ldots, 10\}$.

1.2 The product of two numbers, neither of which is divisible by 10, is 1000. Determine the sum of these numbers.

Hint: First consider the prime factorisation of 1000. Now what can we say about the prime factorisation of our two numbers?

1.3 Find all natural numbers n, such that n is a divisor of $n^2 + 3n + 27$.

Hint: Visibly n divides the first two terms of $n^2 + 3n + 27$. What about the third term?

Advanced

1.4 Show that:

(a) $5 \cdot 17 \mid 5^2 \cdot 17 + 3 \cdot 5 \cdot 9 + 5 \cdot 3 \cdot 8$

(b) $n(n + m) \mid 3mn^2 + amn^2 + 3n^3 + an^3$

Hint: Simplify the right-hand sides.

1.5 Find three three-digit natural numbers whose decimal representations uses nine different digits, and such that their product ends with four zeros.

Hint: A number ending with four zeros is divisible by 10000. Consider the prime factorisation of 10000.

1.6 (a) Find all natural numbers who have exactly 41 divisors and that are divisible by 41.

(b) Find all natural numbers who have exactly 42 divisors and that are divisible by 42.

Hint: First remember how to find the number of divisors, then argue with combinatorics.

Olympiad

1.7 Find all natural numbers n such that $n + 1 \mid n^2 + 1$.

Hint: Try to reduce the degree of n on the right-hand side.

1.8 Show that for all natural numbers n, there are n consecutive natural numbers such that none of them are prime.

Hint: Try finding a number that is divisible by all of $\{1, 2, \dots, (n+1)\}$.

1.9 Show that there are infinitely many natural numbers n, such that $2n$ is a square number, $3n$ a cube, and $5n$ a fifth power.

Hint: Try to construct such a number, then use its structure to find the others.

2 gcd and lcm

Beginner

2.1 (IMO 59) Show that for all natural numbers n, the following fraction is irreducible:

$$\frac{21n+4}{14n+3}$$

Hint: Use Euclid's algorithm to calculate $\gcd(21n+4, 14n+3)$.

2.2 Find all pairs of natural numbers (a, b) such that:

$$\operatorname{lcm}(a, b) = 10 \gcd(a, b)$$

Hint: Let $d = \gcd(a, b)$ and let's write $a = dm$, $b = dn$ where $\gcd(m, n) = 1$. Then we have that $\operatorname{lcm}(a, b) = dmn$. Replace this in the original equation.

Advanced

2.3 Show that every natural number $n > 6$ is the sum of two coprime natural numbers greater than one.

Hint: Separate the problem into an even and an odd case for n. Then, in both instances, try to express n explicitly as such a sum.

2.4 Two natural numbers a and b are said to be *friends* if $a \cdot b$ is a square number. Show that if a and b are friends, then so are a and $\gcd(a, b)$.

Hint: Let $d = \gcd(a, b)$ and let's write $a = dm$, $b = dn$ where $\gcd(m, n) = 1$. When is the product to two coprime numbers a perfect square?

Olympiad

2.5 Let m and n be two natural numbers whose sum is a prime number. Show that m and n are coprime.

Hint: Make an indirect proof: suppose that m and n aren't coprime and try to prove that then their sum isn't prime.

2.6 (Canada 97) Find all pairs of natural numbers (x, y) where $x \leq y$ and such that they satisfy the following equations:

$$\gcd(x, y) = 5! \text{ and } \operatorname{lcm}(x, y) = 50!$$

Hint: Take a closer look at the prime factors of 50!.

3 Estimations

Beginner

3.1 A rectangle is said to be *beautiful* if the lengths of all of its sides are natural numbers, and if the measures of its perimeter and area are equal. Find all the *beautiful* rectangles.

Hint: Let a, b be the sides of the rectangle. When the sides a and b increase, who gets bigger faster: the area or the perimeter?

3.2 Find all pairs of natural numbers (x, y) such that:

$$\frac{1}{x} + \frac{2}{y} = 1.$$

Hint: What happens when both x and y are large? Using this, try to find an upper bound for x and y.

Advanced

3.3 A rectangular parallelepiped is said to be *beautiful* if the lengths of all of its sides are natural numbers, and if the measures of its volumes and surface area are equal.Find all the *beautiful* rectangular parallelepipeds.

Hint: Use exercise **??** as a guide.

3.4 Find all triplets of natural numbers (x, y, z) such that:

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1.$$

Hint: Use exercise **??**.

3.5 Find all natural numbers n such that $n^2 + 1$ is a divisor of $n^7 + 13$.

Hint: Start with $n^2 + 1 \mid n^7 + 13$ and try to reduce the degree of n on the right-hand side.

Olympiad

3.6 Show that the equation

$$y^2 = x(x+1)(x+2)(x+3)$$

has no solution in the natural numbers.

Hint: Try to wedge the right-hand side between two consecutive square numbers.

3.7 Find all integers x for which

$$x! = x^2 + 11x - 36$$

Hint: As x increases, the left-hand side grows much faster than the right-hand side. For which values x is the left-hand side strictly larger than the right-hand side?

3.8 (IMO 98) Find all pairs of natural numbers (a, b) such that $a^2b + a + b$ is divisible by $ab^2 + b + 7$.