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Number Theory I - Exercises

1 Divisibility

Beginner

1.1 Show that 900 divides 10!.

1.2 The product of two numbers, neither of which is divisible by 10, is 1000. Determine the sum of these numbers.

1.3 Find all natural numbers n, such that n is a divisor of $n^2 + 3n + 27$.

Advanced

1.4 Show that:

(a) $5 \cdot 17 \mid 5^2 \cdot 17 + 3 \cdot 5 \cdot 9 + 5 \cdot 3 \cdot 8$

(b) $n(n + m) \mid 3mn^2 + amn^2 + 3n^3 + an^3$

1.5 Find three three-digit natural numbers whose decimal representations uses nine different digits, and such that their product ends with four zeros.

1.6 (a) Find all natural numbers who have exactly 41 divisors and that are divisible by 41.

(b) Find all natural numbers who have exactly 42 divisors and that are divisible by 42.

Olympiad

1.7 Find all natural numbers n such that $n + 1 \mid n^2 + 1$.

1.8 Show that for all natural numbers n, there are n consecutive natural numbers such that none of them are prime.

1.9 Show that there are infinitely many natural numbers n, such that $2n$ is a square number, $3n$ a cube, and $5n$ a fifth power.

2 gcd and lcm

Beginner

2.1 (IMO 59) Show that for all natural numbers n, the following fraction is irreducible:

$$\frac{21n+4}{14n+3}$$

2.2 Find all pairs of natural numbers (a, b) such that:

$$\mathrm{lcm}(a, b) = 10\,\gcd(a, b)$$

Advanced

2.3 Show that every natural number $n > 6$ is the sum of two coprime natural numbers greater than one.

2.4 Two natural numbers a and b are said to be *friends* if $a \cdot b$ is a square number. Show that if a and b are friends, then so are a and $\gcd(a, b)$.

Olympiad

2.5 Let m and n be two natural numbers whose sum is a prime number. Show that m and n are coprime.

2.6 (Canada 97) Find all pairs of natural numbers (x, y) where $x \leq y$ and such that they satisfy the following equations:

$$\gcd(x, y) = 5! \text{ and } \mathrm{lcm}(x, y) = 50!$$

3 Estimations

Beginner

3.1 A rectangle is said to be *beautiful* if the lengths of all of its sides are natural numbers, and if the measures of its perimeter and area are equal. Find all the *beautiful* rectangles.

3.2 Find all pairs of natural numbers (x, y) such that:

$$\frac{1}{x} + \frac{2}{y} = 1.$$

Advanced

3.3 A rectangular parallelepiped is said to be *beautiful* if the lengths of all of its sides are natural numbers, and if the measures of its volumes and surface area are equal.Find all the *beautiful* rectangular parallelepipeds.

3.4 Find all triplets of natural numbers (x, y, z) such that:

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1.$$

3.5 Find all natural numbers n such that $n^2 + 1$ is a divisor of $n^7 + 13$.

Olympiad

3.6 Show that the equation

$$y^2 = x(x+1)(x+2)(x+3)$$

has no solution in the natural numbers.

3.7 Find all integers x for which

$$x! = x^2 + 11x - 36$$

3.8 (IMO 98) Find all pairs of natural numbers (a, b) such that $a^2b + a + b$ is divisible by $ab^2 + b + 7$.

4 Previous Olympiad Problems

The best preparation is going through past papers. Don't check the solutions too quickly though!

(Preliminary Round 2012, 1) Find all pairs of natural numbers (m, n) such that $(m+1)(n+2)$ is divisible by mn.

(Preliminary Round 2004, 1) Find all natural numbers a, b et n such that:

$$a! + b! = 2^n$$

(Preliminary Round 2005, 3) Let m, n be two coprime integers. Show that the two integers $m^3 + mn + n^3$ and $mn(m+n)$ are also coprime.

(Preliminary Round 2011, 2) Find all natural numbers n such that n^3 is the product lf all the positive divisors n.

(Preliminary Round 2006, 1) Find all triplets of prime numbers (p, q, r) such that their differences

$$\mid p - q \mid, \mid q - r \mid, \mid r - p \mid$$

are also prime numbers.

(Preliminary Round 2008, 4) Find all natural numbers n such that the number of divisors of n is equal to the third smallest divisor of n.

(Preliminary Round 2013, 4) Find all pairs of natural numbers (m, n) such that

$$(m+1)! + (n+1)! = m^2n^2$$