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Geometry I - Hints

1 Angles in a triangle

Beginner

1.1 Let ABC be a triangle with $AB = AC$ in which the angle bisector of $\angle ABC$ is perpendicular to AC. Show that ABC is an equilateral triangle.

Hint: Calculate the angles at B and C.

Advanced

1.2 Let ABC be a triangle with $AB > AC$. The angle bisector of the exterior angle at C intersect the angle bisector of $\angle ABC$ in D. The parallel to BC through D intersect CA in L and AB in M.
Show that $LM = BM - CL$ holds.

Hint: Show that $BM = DM$ and $CL = DL$.

2 Angles in a circle

Beginner

2.1 The points A, B, C and D lie on a circle in this order. Calculate the angle $\angle DBA$ given:

(a) $\angle DCA = 56° \implies \mathbf{56°}$

(b) $\angle CBD = 39°$, $\angle ADC = 121° \implies \mathbf{20°}$

(c) $\angle CBA = 91°$, $\angle CAD = 13° \implies \mathbf{78°}$

(d) $\angle ADB = 41°$, $\angle DCB = 103° \implies \mathbf{62°}$

(e) $\angle BAD = 140°$, $\angle ACB = 17° \implies \mathbf{23°}$

2.2 Let ABC be a triangle and P the intersection of the angle bisector of $\angle BAC$ and the circumcircle of the triangle ABC. Show that BPC is an isosceles triangle.

Hint: What can be said about the angles $\angle PBC$ and $\angle BCP$ using the Inscribed Angle Theorem?

2.3 Let $ABCD$ be a quadrilateral with $\angle BAD = 131°$, $\angle DBA = 17°$ and $\angle ACB = 32°$. What is the size of $\angle DCA$?

Hint: First try to prove that $ABCD$ is a cyclic quadrilateral.

2.4 Let ABC be a triangle with circumcircle k and circumcircle center O. Denote by t the tangent to k in A. Let s be the reflection of the line AB at t. Show that s is a tangent to the circumcircle of the triangle ABO.

Hint: Try to show that the angle enclosed by s and AB is equal to $\angle AOB$. The statement then follows with the inversion of the Tangent-chord Theorem.

2.5 Let A and B be two distinct points and k be the circle with diameter AB. Argue why the inscribed angles over the line AB are all $90°$. (Such a circle is called the *Thales circle* over the distance AB).

Hint: Use the Central Angle Theorem (What is the size of the central angle?).

Advanced

2.6 Let ABC be a triangle with incircle center I. The line CI intersects the circumcircle of the triangle ABI a second time in D and AI intersects the circumcircle of the triangle BCI a second time in E.
Show that the points D, E und B are collinear.

Hint: Determine the angles $\angle ABD$ and $\angle EBC$ depending on α, β and γ (defined as usual) and then show that $\angle EBC + \angle CBA + \angle ABD = 180°$ holds.

2.7 Let ABC be a right-angled triangle and M the midpoint of the hypotenuse AB. Show that $AM = BM = CM$ holds.

Hint: Observe that C lies on the Thales circle above AB. It follows that M is the center of this circle. Why does $AM = BM = CM$ follow?

Olympiad

2.8 In the right-angled triangle ABC let M be the midpoint of the hypotenuse AB, H the foot of the altitude through C and W the intersection of AB with the angle bisector of $\angle ACB$. Show that $\angle HCW = \angle WCM$ holds.

Hint: Use 2.7 and consider how the statement can be expressed in angles.

2.9 The medians AA', BB' and CC' in triangle ABC intersect the circumcircle of triangle ABC further times at points A_0, B_0 and C_0 respectively (this formulation automatically implies that A', B' and C' are the corresponding side centers of triangle ABC). Assume that the centroid S bisects the distance AA_0. Show that then $A_0B_0C_0$ is an isosceles triangle.

Hint: It is known that the centroid divides the medians in the ratio 2:1. So $SA_0 = AS = 2SA'$ holds, consequently A' bisects the distance SA_0. Now $SA' = A'A_0$ and $A'B = A'C$, therefore SBA_0C is a parallelogram. Now try to express the angles $\angle C_0B_0A_0$ and $\angle A_0C_0B_0$ depending on the angles in the parallelogram (Inscribed Angle Theorem!).

3 Cyclic quadrilaterals

Beginner

3.1 Let ABC be a triangle with orthocenter H and H_A, H_B and H_C being the feet of the altitudes. Show that AH_CHH_B and BCH_BH_C are cyclic quadrilaterals.

Hint: Find two opposite angles each that complete to 180°.

Advanced

3.2 Let ABC be a triangle and let D, E and F be points on the sides BC, CA and AB respectively. Denote by P the second intersection of the circumcircles of the triangles FBD and DCE. Show that $AFPE$ is a cyclic quadrilateral.

Hint: Show $\angle PFA = \angle PDB$ and $\angle PDB = \angle PEC$.

3.3 et $ABCD$ be a rectangle and M the midpoint of the side AB. Let P be the projection of C onto the line MD (which means P lies on MD such that CP and MD are perpendicular to each other).
Show, that PBC is an isosceles triangle.

Hint: Show that $MBCP$ is a cyclic quadrilateral and use $\angle DMA = \angle BMC$.

3.4 Let ABC be a triangle with orthocenter H and D, E and F being the feet of the altitudes. Show that H is the center of the incircle of the triangle DEF.

Hint: Find as many cyclic quadrilaterals as possible and then show $\angle EDH = \angle HDF$ using the Inscribed Angle Theorem (applied in multiple cyclic quadrilaterals).

3.5 Let $ABCD$ be a convex quadrilateral in which the diagonals are perpendicular to each other (*convex* means for n-corners that all interior angles are $\leq 180°$). Denote by P the diagonal intersection. Show that the four projections of P onto the lines AB, BC, CD and DA form a cyclic quadrilateral.

Hint: Let E, F, G and H be the projections on the lines AB, BC, CD and DA. Find as many cyclic quadrilaterals as possible and then show $\angle FEP = \angle 90° - \angle PGF$ and $\angle PEH = 90° - \angle HGP$.

3.6 Let ABC be a triangle with orthocenter H. Let M be the midpoint of the line AH and N the midpoint of the line BC.
Show that the three feet of the altitudes lie on the Thales circle with regard to MN.
Why does this imply that the three feet of the altitudes, the three side midpoints, and the midpoints of the lines AH, BH, and CH all lie on a circle? (This circle is called the *Feuerbach's circle* or *nine-point circle*).

Hint: Let D, E and F be the feet of the altitudes on the sides BC, CA and AB. Because of $\angle MDN = 90°$ D lies on the Thales circle above MN. For E and F there is more to do. First establish that E and F lie on the Thales circles over AH and BC. M and N are the midpoints of these Thales circles, so $AM = HM = EM = FM$ and $BN = CN = EN = FN$ holds. Now the following holds:

$$\angle NFC + \angle CFN = \angle FCN + \angle MHF = \angle HCD + \angle DHC = 180° - \angle CDH = 90°.$$

For the second part: We have shown that M and N lie on the circumcircle of the triangle EDF. In exactly the same way we can show that the midpoints of CA and AB as well as the midpoints of the distances BH and CH lie on this circumcircle.

Olympiad

3.7 Let A and B be two distinct points on the circle k. Let the point C lie on the tangent to k through B and let $AB = AC$. Let the intersection point of the angle bisector of $\angle ABC$ with AC be D. Assume that the point D lies inside k. Show that $\angle ABC > 72°$ holds.

Hint: Let $\angle ABC = \beta$. Let P be any point on the circle k which lies on the other side of AB than C. According to the Tangent-chord Theorem $\angle APB = \beta$. Now let Q be a point on the circle k that lies on the same side of AB as C. $AQBP$ is a cyclic quadrilateral, so $\angle BQA = 180° - \beta$ holds. Now note that D lies inside k exactly when $\angle BDA > \angle BQA$ holds. This is equivalent to $\frac{3}{2}\beta > 180° - \beta$, since $\angle BDA = \angle DBC + \angle BCD = \frac{3}{2}\beta$. Transforming the inequality now yields $\beta > 72°$.

3.8 Two circles k_1 and k_2 with centers M_1 and M_2, respectively, intersect at points A and B. The line M_1B intersects k_2 in $F \neq B$ and M_2B intersects k_1 in $E \neq B$. The parallel to EF through B intersect k_1 and k_2 in the points P and Q, respectively.

(a) Show that B is the center of the incircle of the triangle AEF.

(b) Show that $PQ = AE + AF$ holds.

Hint:

(a) Observe that $M_1A = M_1B = M_1E$ and $M_2A = M_2B = M_2F$ hold and that the Central Angle Theorem can be applied. Now show with angle chasing that M_1AFE and AM_2FE are cyclic quadrilaterals (it follows immediately that M_1, A, M_2, F and E lie on a circle). The rest is again angle chasing.

(b) Let S be the intersection point of PB and AE. Show that PAS and SBE are isosceles triangles. It follows that $PB = AE$ holds. Show $BQ = AF$ analogously.