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Exercises Geometry I

1 Angles in a triangle

Beginner

1.1 Let ABC be a triangle with $AB = AC$ in which the angle bisector of $\angle ABC$ is perpendicular to AC. Show that ABC is an equilateral triangle.

Advanced

1.2 Let ABC be a triangle with $AB > AC$. The bisector of the exterior angle at C intersects the angle bisector of $\angle ABC$ in D. The parallel to BC through D intersects CA in L and AB in M. Show, that $LM = BM - CL$ holds.

2 Angles in a circle

Beginner

2.1 The points A, B, C and D lie on a circle in this order. Calculate the angle $\angle DBA$ given:

(a) $\angle DCA = 56°$.

(b) $\angle CBD = 39°$, $\angle ADC = 121°$.

(c) $\angle CBA = 91°$, $\angle CAD = 13°$.

(d) $\angle ADB = 41°$, $\angle DCB = 103°$.

(e) $\angle BAD = 140°$, $\angle ACB = 17°$.

2.2 Let ABC be a triangle and P the intersection point of the angle bisector of $\angle BAC$ and the circumcircle of the triangle ABC. Show that BPC is an isosceles triangle.

2.3 Let $ABCD$ be a quadrilateral with $\angle BAD = 131°$, $\angle DBA = 17°$ and $\angle ACB = 32°$. What is the size of $\angle DCA$?

2.4 Let ABC be a triangle with circumcircle k and circumcircle center O. Denote by t the tangent to k in A. Let s be the reflection of the line AB at t. Show that s is a tangent to the circumcircle of the triangle ABO.

2.5 Let A and B be two distinct points and k be the circle with diameter AB. Argue why the inscribed angles over the line AB are all $90°$. (Such a circle is called the *Thales circle* over the segment AB).

Advanced

2.6 Let ABC be a triangle with incircle center I. The line CI intersects the circumcircle of the triangle ABI one more time in D and AI intersects the circumcircle of the triangle BCI one more time in E.
Show that the points D, E and B are collinear.

2.7 Let ABC be a right-angled triangle and M the center of the hypotenuse AB. Show that $AM = BM = CM$ holds.

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2.8 In the right-angled triangle ABC let M be the center of the hypotenuse AB, H the foot of the altitude through C and W the intersection of AB with the angle bisector of $\angle ACB$. Show that $\angle HCW = \angle WCM$ holds.

2.9 The medians AA', BB' and CC' in triangle ABC intersect the circumcircle of triangle ABC again at points A_0, B_0 and C_0 respectively (this formulation automatically implies that A', B' and C' are midpoints of the sides of triangle ABC). Assume that the centroid S bisects the distance AA_0. Show then that $A_0B_0C_0$ is an isosceles triangle.

3 Cyclic quadrilaterals

Beginner

3.1 Let ABC be a triangle with orthocenter H and H_A, H_B and H_C being the feet of the altitudes. Show that AH_CHH_B and BCH_BH_C are cyclic quadrilaterals.

Advanced

3.2 Let ABC be a triangle and let D, E and F be points on the sides BC, CA and AB respectively. Denote by P the second intersection of the circumcircles of the triangles FBD and DCE. Show that $AFPE$ is a cyclic quadrilateral.

3.3 Let $ABCD$ be a rectangle and M the midpoint of the side AB. Let P be the projection of C onto the line MD (which means P lies on MD such that CP and MD are perpendicular to each other).
Show, that PBC is an isosceles triangle.

3.4 Let ABC be a triangle with orthocenter H and D, E and F being the feet of the altitudes. Show that H is the center of the incircle of the triangle DEF.

3.5 Let $ABCD$ be a convex quadrilateral in which the diagonals are perpendicular to each other (*convex* means for n-corners that all interior angles are $\leq 180°$). Denote by P the diagonal intersection. Show that the four projections of P onto the lines AB, BC, CD and DA form a cyclic quadrilateral.

3.6 Let ABC be a triangle with orthocenter H. Let M be the midpoint of the line AH and N the midpoint of the line BC.
Show that the three feet of the altitudes lie on the Thales circle with regard to MN.
Why does this imply that the three feet of the altitudes, the three side midpoints, and the midpoints of the lines AH, BH, and CH all lie on a circle? (This circle is called the *Feuerbach's circle* or *nine-point circle*).

Olympiad

3.7 Let A and B be two distinct points on the circle k. Let the point C lie on the tangent to k through B and let $AB = AC$. Let the intersection point of the angle bisector of $\angle ABC$ with AC be D. Assume that the point D lies inside k. Show that $\angle ABC > 72°$ holds.

3.8 Two circles k_1 and k_2 with centers M_1 and M_2, respectively, intersect at points A and B. The line M_1B intersects k_2 in $F \neq B$ and M_2B intersects k_1 in $E \neq B$. The parallel to EF through B intersects k_1 and k_2 in the points P and Q, respectively.

(a) Show that B is the center of the incircle of the triangle AEF.

(b) Show that $PQ = AE + AF$ holds.

4 Problems from previous Olympiads

Old exam problems are very suitable for preparation; on the one hand, they naturally correspond to the exam level, and on the other hand, all solutions to the problems can be found on the homepage `www.mathematical.olympiad.ch`. However, one should always have worked on the problems oneself first before looking at the sample solutions to them!

1. **(Preliminary Round 2010, 2.)** Let g be a straight line in the plane. Circles k_1 and k_2 lie on the same side of g and touch g at points A and B respectively. Let another circle k_3 touch k_1 in D and k_2 in C. Prove that the following holds:

   (a) The quadrilateral $ABCD$ is cyclic.

   (b) The lines BC und AD intersect on k_3.

2. **(Preliminary Round 2011, 1.)** Let ABC be a triangle with $\angle CAB = 90°$. The point L lies on the side BC. The circumcircle of the triangle ABL intersects the line AC in M and the circumcircle of the triangle CAL intersects the line AB in N. Assume N lies inside the side AB and M is on the extension of the side AC. Show that L, M and N lie on a straight line.

3. **(Preliminary Round 2012, 3.)** Let A and B be the intersection points of two circles k and l with centers K and L, respectively. Let M and N be the intersection points of k and l with a

straight line through A, so that A lies between M and N. Let D be the intersection of the lines MK and NL. Show that the points M, N, B and D lie on a circle.

4. **(Preliminary Round 2013, 2.)** Let M_1 and M_2 be the centers of the circles k_1 and k_2, respectively, which intersect perpendicularly at the point P. Furthermore, let k_1 intersect the line M_1M_2 in Q. Show that the perpendicular to the line M_1M_2 through the point M_2 and the straight line PQ intersect at k_2.

5. **(Preliminary Round 2014, 2.)** Two circles k_1, k_2 with centers M_1 and M_2, respectively, intersect at points A and B. The tangent to k_1 through A intersects k_2 one more time at point P, while the straight line M_1B intersects k_2 one more time at point Q. Assume that Q lies outside k_1 and that $P \neq Q$ holds. Show that PQ is parallel to M_1M_2.