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# Geometry I - Angle Chasing

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## Contents

# 1 Introduction

This script is intended to give an initial overview of the geometry problems as they are set at the Mathematical Olympiads. The theory needed to solve the problems is neither too much nor too difficult to understand, but rather quite intuitive. The difficulty of the tasks is rather in finding a reasonable approach, recognizing the essential elements in a sketch and linking the facts found in such a way that a complete proof emerges.
It is assumed that the reader is familiar with the construction of geometric figures (e.g. the Thales circle). In particular, the existence, construction and most important properties of the following special points in the triangle should be known: center of the circumcircle, center of the incircle, intersection of the altitudes, intersection of the medians (which of these mentioned points can also lie outside of the triangle? What does the triangle look like then?). Additionally, it is expected that elementary theorems like the Pythagorean theorem and basic concepts such as acute angles ($< 90°$) and obtuse angles (between $90°$ and $180°$) are known.
In each chapter there are some examples with solutions. They are meant to get you into the topic and show the given theorems in action. The solutions of the examples are rather thorough. In the exams you do not have to give the proofs in such a detailed mannner, but you must not miss any steps or cases (see chapter 5).

# 2 Angles in the triangle

We dive right in with a first example.

**Example 1** *Prove that in an arbitrary triangle the sum of the interior angles is 180 degrees.*

*Solution.* As usual let $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$ and let $g$ be the parallel line to $AB$ through the point $C$ (Fig. 1). Since the three angles at $C$ form together a straight line, the following applies

$$\angle(g, CA) + \gamma + \angle(CB, g) = 180°.$$

In addition $\angle(g, CA)$ is equal to $\alpha$ and $\angle(CB, g) = \beta$, since they form alternate angles at the parallels $g$ and $AB$. This gives $\alpha + \beta + \gamma = 180°$. □

Since any convex $n$-polygon can be constructed from $n - 2$ triangles, the interior angle sum for any $n$-polygon is $(n - 2) \cdot 180°$ (the exact proof can be stated with induction).
From the example 1 follows the **Exterior Angle Theorem** immediately, which is best made clear to oneself by drawing a sketch.

**Proposition 2.1** (Exterior Angle Theorem) *An exterior angle of a triangle is as big as the sum of the two non-adjacent interior angles.*

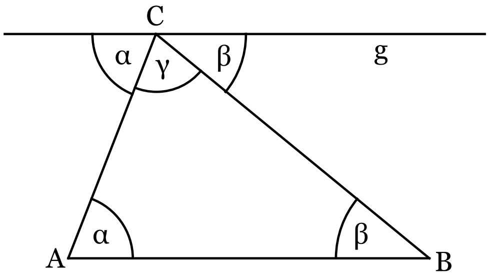


Figure 1: Solution to example 1

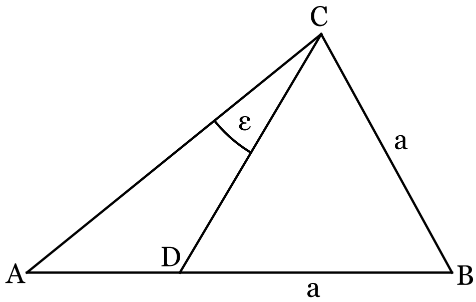


Figure 2: Example 2

**Example 2** *In the triangle $ABC$ let the side $AB$ be bigger than the side $BC$. Let the point $D$ lie on the line $AB$, such that $BD = BC$ (Fig. 2). How big is $\epsilon = \angle ACD$, if you know, that $\angle BCA - \angle CAB = 30°$ ?*

*Solution.* Again let $\alpha$ and $\gamma$ be the angles of $\triangle ABC$ at $A$ and $C$, respectively. The angle $\angle CDB$ is the exterior angle of $\triangle ADC$ and thus as big as the sum of the two interior angles at $A$ and $C$ (Fig. 3):

$$\angle CDB = \alpha + \epsilon. \tag{1}$$

According to the condition, $\triangle BCD$ is isosceles and thus the following holds

$$\angle CDB = \angle BCD = \gamma - \epsilon. \tag{2}$$

Combining the equations (1) and (2) leads to

$$\begin{aligned} \alpha + \epsilon = \angle CDB = \gamma - \epsilon \\ \Leftrightarrow \alpha + 2\epsilon = \gamma \\ \Leftrightarrow 2\epsilon = \gamma - \alpha = 30° \\ \Leftrightarrow \epsilon = 15° \end{aligned}$$

□

The constraint showcased in the last example seems quite strange at first sight and one does not really know what to do with it. This is often the case and there is no foolproof system for how to proceed in each case. In this task we were successful by simply ignoring a condition at the beginning. Only after we had established a relationship between the angles mentioned in the task $(\alpha, \gamma, \epsilon)$ did it become clear how to use the constraint. But sometimes it is also the case that one can unearth hints for the solution of the problem from the secondary conditions.

**Example 3** *Let $ABC$ be a right-angled triangle with hypotenuse $AB$ (Fig. 4). Let the point $P$ have the property that the the lines $PB$ and $AB$ are perpendicular to each other and that $PB = CB$ holds. Show that the line $PC$ is either perpendicular or parallel to the angle bisector $w_\alpha$ of the angle $\alpha = \angle CAB$.*

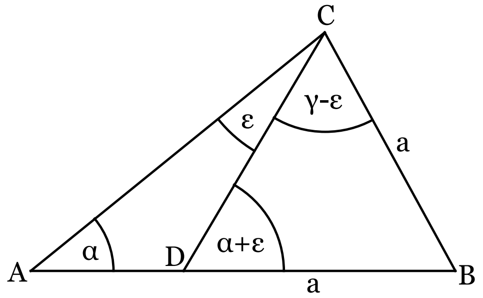


Figure 3: Solution to example 2

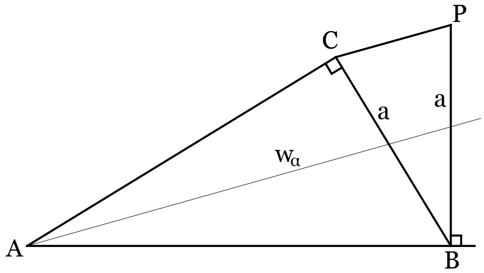


Figure 4: Example 3

*Solution.* There are two points which satisfy the conditions for $P$. Let $P_1$ be the possibility which lies on the same side of the line $AB$ as $C$ (Fig. 5) and $P_2$ be the possibility which lies on the opposite side of $AB$ (Fig. 6).
Since $\gamma = 90°$, there is a direct relationship between $\alpha$ and $\beta$:

$$\beta = 90° - \alpha$$

and thus

$$\angle CBP_1 = \alpha.$$

The triangle $BP_1C$ is isosceles according to the prerequisite and therefore the following applies:

$$\angle BP_1C = \angle P_1CB.$$

For the sum of angles in this triangle we get

$$\begin{aligned} \alpha + 2 \cdot \angle P_1CB &= 180° \\ \Leftrightarrow \quad \angle P_1CB &= 90° - \frac{\alpha}{2}. \end{aligned}$$

Let $Q$ be the intersection point of $w_\alpha$ and $CB$. Since $\angle CAQ = \frac{\alpha}{2}$, we get

$$\angle AQC = 90° - \frac{\alpha}{2} = \angle P_1CB.$$

$\angle AQC$ and $\angle P_1CB$ are also alternate angles and therefore $w_\alpha$ and $P_1C$ are parallel.
Let us now consider the situation for $P_2$. It holds that $\angle P_2BC = 180° - \alpha$ and thus

$$\angle BCP_2 = \frac{\alpha}{2}.$$

Let now $R$ be the intersection point of $w_\alpha$ with $CP_2$. In $\triangle QCR$ it holds that

$$\angle QCR + \angle RQC = \angle BCP_2 + (90° - \angle CAQ) = \frac{\alpha}{2} + 90° - \frac{\alpha}{2} = 90°.$$

Thus $\angle CRQ = 90°$ and $w_\alpha \perp CP_2$. $\square$

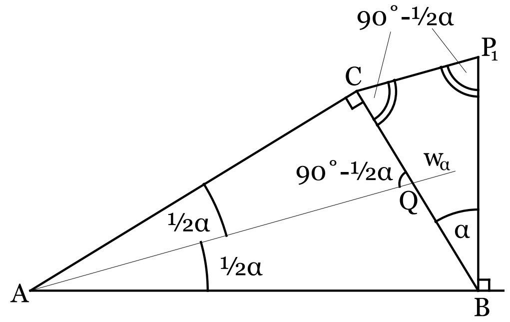


Figure 5: Solution to example 3, Case 1

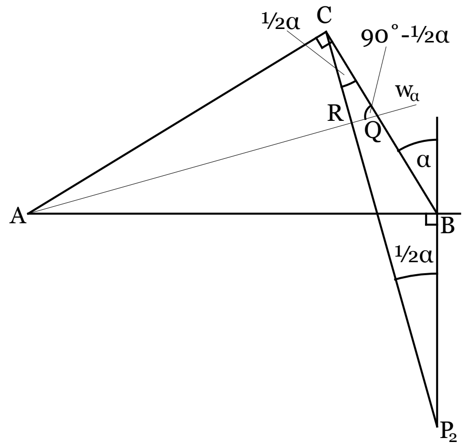


Figure 6: Solution to example 3, Case 2

This example shows nicely what angle chasing actually is. At the beginning an angle is replaced with a placeholder variable (here: $\angle CAB \doteq \alpha$) and then all further angles in the figure are expressed in dependence of this angle (e.g. $\angle AQC = 90° - \frac{\alpha}{2}$) - which means the whole figure is determined if $\alpha$ is determined. In other problems it is sometimes necessary to introduce several variable.

# 3 Angles in a circle

Let $O$ be the center of a circle and let $A$ and $B$ be two points lying on the circle line. The angle $AOB$ is called **center angle** over the arc $\widehat{AB}$. Note that there are always two possibilities for $AB$, one is $\angle AOB \leq 180°$ and the other is $\angle AOB \geq 180°$. Usually it is clear from the context which of the two possibilities is meant, but it is better to declare this unambiguously (as in the following sentence).

**Proposition 3.1** (Inscribed-Central Angle Theorem) *Let $O$ be the circumcenter of any triangle $ABC$ with $\angle BAC = \alpha$. The central angle of the arc $BC$, which does not contain the point $A$, is then equal to $2\alpha$.*

*Proof.* We prove the case when $\triangle ABC$ is acute-angled (Fig. 7); the other case works analogously. Because $\triangle ABC$ is acute-angled, $O$ lies inside $\triangle ABC$ and we can split $\angle BAC$ into two smaller angles by setting $\alpha_1 = \angle OAB$ and $\alpha_2 = \angle CAO$. Since $O$ is the center of the circumcircle, $\triangle OAB$ and $\triangle OCA$ are isosceles, which means

$$\angle ABO = \alpha_1 \quad \angle OCA = \alpha_2.$$

If we extend the straight line $AO$ and intersect it with the circumcircle in $D$, we see that $\angle DOB$ is the exterior angle of $\triangle OAB$ and $\angle COD$ is the exterior angle of $\triangle OCA$.

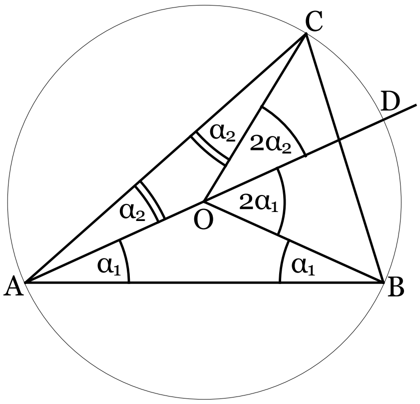


Figure 7: Inscribed-Central Angle Theorem

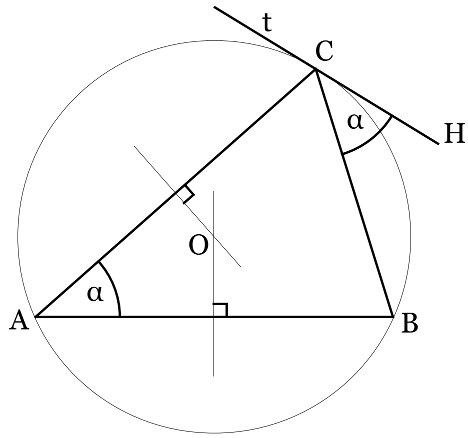


Figure 8: Tangent-chord Theorem

Therefore we get, according to the Exterior Angle Theorem

$$\angle COB = \angle COD + \angle DOB = 2\alpha_2 + 2\alpha_1 = 2(\alpha_1 + \alpha_2) = 2\alpha.$$

□

Analogously to the central angle, the angle $BAC$ in Fig. 7 called **inscribed angle** is above the arc $\widehat{BC}$. Consider what happens to the magnitude of the inscribed angle when you move the point $A$ around on the circumcircle. This will take you directly to the next theorem.

**Proposition 3.2** (Inscribed Angle Theorem) *Let $BC$ be any arc on a circle $k$. All inscribed angles over $BC$ are equal.*

*Proof.* We again consider Fig. 7 and choose the arc $BC$ containing $D$. The center angle $COB$ is obviously independent of the choice of $A$, thus it is clear that also the magnitude of $\angle CAB$ must remain constant as long as $A$ does not change sides with regard to the line $BC$. □

**Proposition 3.3** (Tangent-chord Theorem) *In any triangle $ABC$ let $k$ be the circumcircle with center $O$. Let $t$ be the tangent in $C$ to the circle $k$ (Fig. 8). $C$ divides $t$ into two half-straights; choose an arbitrary auxiliary point $H$ on that half-straight line which lies on the other side of $BC$ than $A$. It holds $\angle BAC = \angle BCH$.*

*Proof.* (Fig. 9) With the Inscribed Angle Theorem we calculate

$$\angle BOC = 2\alpha.$$

$\triangle OBC$ is isosceles and we can therefore calculate the angle $OCB$ with the help of the interior angle sum:

$$\angle OCB = \frac{180° - 2\alpha}{2} = 90° - \alpha.$$

Tangents are always perpendicular to the respective radius, which leads to $\angle OCH = 90°$. The last step is now no longer difficult:

$$\angle BCH = 90° - \angle OCB = 90° - (90° - \alpha) = \alpha.$$

□

The Tangent-chord Theorem is very useful for complicated problems, because it allows to calculate angles on the circle without the center of the circle being introduced. It keeps the sketch much cleaner. Next, we will look at an illustrative example.

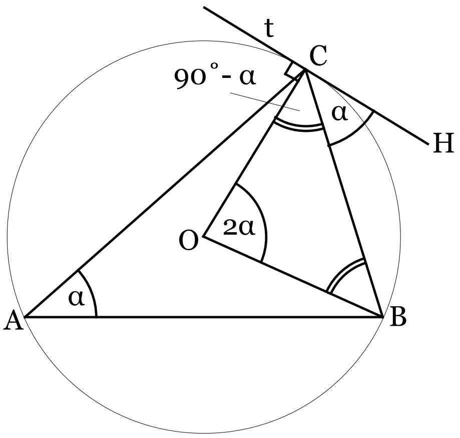


Figure 9: Proof of the Tangent-chord Theorem

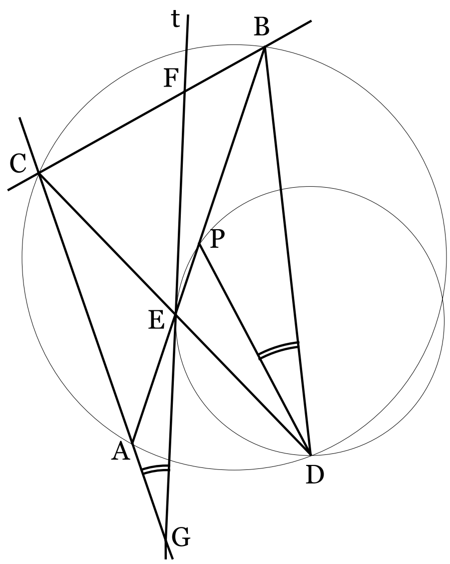


Figure 10: Example 4

**Example 4** *The two chords $AB$ and $CD$ intersect inside the circle at $E$ (Fig. 10). Let $P$ be any point on the line $BE$. The tangent $t$ to the circle through $D$, $P$ and $E$ in $E$ intersects the line $BC$ in $F$ and $AC$ in $G$. Show $\angle FGC = \angle BDP$.*

*Solution.* (Abb. 11) Firstly we set

$$\angle EPD = \alpha, \quad \angle ACE = \beta$$

(it does not matter so much which angles we substitute).
Now we can derive further angles. Thus, according to the Tangent-chord theorem

$$\angle GED = \alpha$$

which is an exterior angle of $\triangle ECG$. It follows that

$$\angle FGC = \alpha - \beta.$$

According to the Inscribed Angle Theorem the following holds:

$$\angle ABD = \beta$$

and using the exterior angle $\angle EPD = \alpha$ of $\triangle BDP$ we get

$$\angle BDP = \alpha - \beta$$

thus equal to $\angle FGC$. □

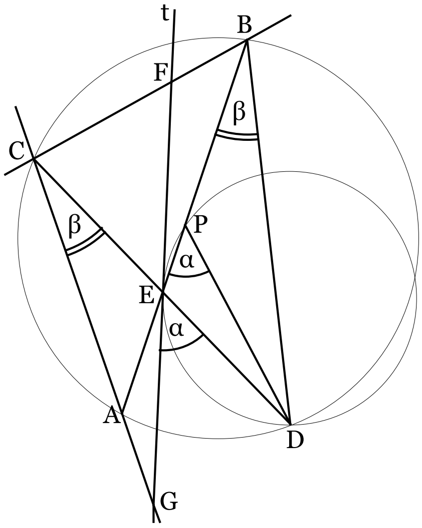


Figure 11: Solution of example 4

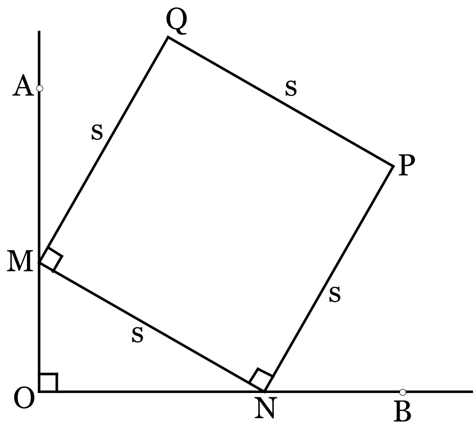


Figure 12: Example 5

The solution of this task looks quite simple, but it is easy to get lost in the many straight lines and circles. Keeping track of the whole situation is one of the main difficulties in geometry. The patience to carefully construct a large sketch with ruler and compass often pays off!
In the solution of this problem we have substituted two angles at the beginning ($\alpha$ and $\beta$). If you want to determine all angles in the figure, you need even more placeholders. Try to determine all angles in the figure as an exercise. How many angles do you have to substitute minimally? (Answer: the figure has four degrees of freedom (if you ignore translation, scaling and rotation); by four substituents you can express all angles present in fig. 11).

**Example 5** *Let $\angle AOB$ be a right angle with a point $M$ on the rey $OA$ and a point $N$ on the ray $OB$ (Fig. 12). Complete $M$ and $N$ to form a square $MNPQ$ such that $P$ is on the other side of $MN$ than $O$. What are the possible locations of the center of the square $MNPQ$, given that the two points $M$ and $N$ can move freely on their rays?*

*Solution.* We first draw the two diagonals of the square and name their intersection $D$ (Fig. 13). At first glance, one would expect $D$ to move on an infinite surface like $P$ and $Q$. We now show that this is not the case.

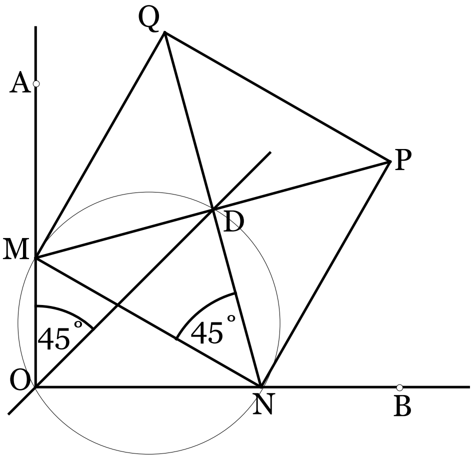


Figure 13: Solution 5, first part

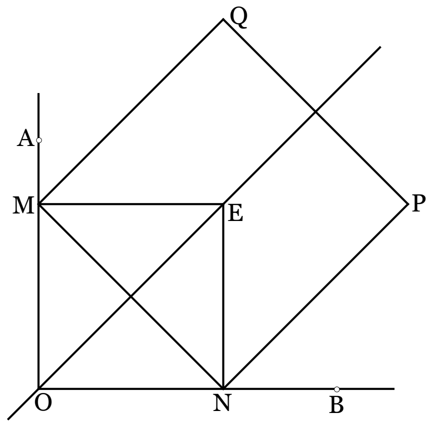


Figure 14: Solution 5, second part

The two angles $\angle MON$ and $\angle MDN$ are both right angles and therefore it makes sense to draw the **thales circle** over the distance $MN$. Now the two points $O$ and $D$ are also on this circle. Since $MNPQ$ is a square, $\angle measuresDNM = 45°$. According to the Inscribed Angle Theorem, $\angle DOM = \angle DNM = 45°$. Thus, the centers of the square lie on the bisector of $\angle AOB$ regardless of the choice of $M$ and $N$.
But we are not finished yet, because it is not proven yet that also all points on the angle bisector can be square center points. To show this, we choose any point $E$ on the angle bisector of $\angle AOB$ and draw the square $MONE$ with $M$ on $OA$ and $N$ on $OB$ (Fig. 14). Now drawing the square $MNPQ$ shows that $E$ is obviously its center. It is thus shown that the center of the square can move on the whole bisector (here a ray) of $\angle AOB$. □

# 4 Cyclic quadrilateral

This brings us to the most important chapter, because cyclic quadrilaterals appear in almost every competition problem. A cyclic quadrilateral is a convex quadrilateral formed by four points lying on a circle. We first examine an important property.

**Proposition 4.1** *In a cyclic quadrilateral $ABCD$ the sum of two opposite interior angles always equals* $180°$.

*Proof.* When dealing with cyclic quadrilaterals, it is usually advisable to draw the two diagonals and divide the interior angles in this way (Fig. 15). We then call $\angle CAD = \alpha$ and $\angle BAC = \beta$. Using the Inscribed Angle Theorem we get

$$\angle CBD = \alpha, \quad \angle BDC = \beta$$

and with the sum of angles in $\triangle BCD$, it follows

$$\angle BCD = 180^\circ - \alpha - \beta = 180^\circ - \angle BAD$$

$\square$

Okay, that was not that spectacular. More surprising is that the reverse of the theorem also applies!

**Proposition 4.2** *Given four points $A, B, C, D$ which form a convex quadrilateral in this order. If $\angle BAD + \angle BCD = 180^\circ$, the four points lie on a circle.*

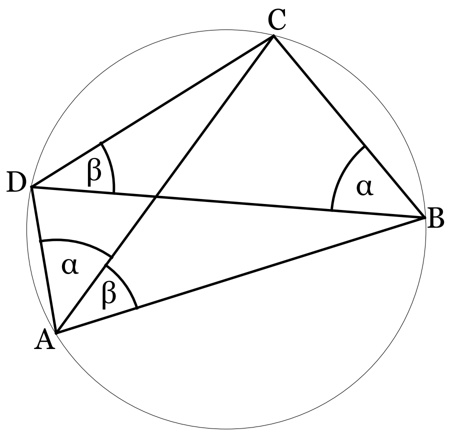


Figure 15: Solution to theorem 5

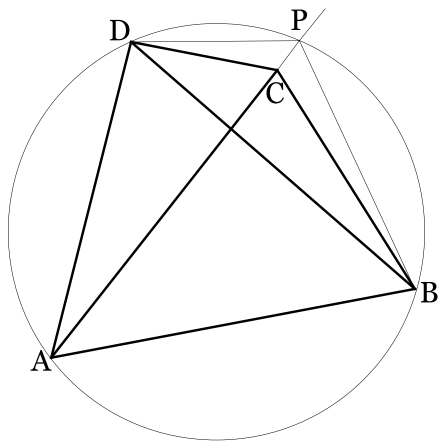


Figure 16: Solution to theorem 6

*Proof.* (Fig. 16) We first draw the circumcircle $k$ of $\triangle ABD$. Assume that $C$ does not lie on $k$. If this assumption leads to a contradiction, the theorem is proved. We first consider the case when $C$ lies inside the circle $k$.
Let the intersection of the line $AC$ with $k$ be $P$. Since $ABPD$ is a cyclic quadrilateral, we know that (theorem 4.1)

$$\angle BAD + \angle BPD = 180^\circ.$$

If we could now show that $\angle BCD$ is greater than $\angle BPD$, we would have the desired contradiction, because then the sum $\angle BAD + \angle BCD$ could not possibly be $180^\circ$, which is given by the premise. Intuitively it is clear that in Fig. 16 $\angle BCD > \angle BPD$ holds, one can prove it like this:
Because $C$ must lie within triangle $BDP$, $\angle CBD < \angle PBD$ and $\angle CDB < \angle PDB$. Thus the sums of the angles give

$$\angle BCD = 180^\circ - \angle CBD - \angle CDB > 180^\circ - \angle PBD - \angle PDB = \angle BPD.$$

The case if $C$ is outside $k$ works completely analogously. $\square$

The argument used here might make an inelegant first impression on you, but in fact the method of the proof even has its own name. The methodology is called 'Working Backward' and occurs quite often. So it is worthwhile to remember the style of approach. Furthermore, we already had a simple example with a cyclic quadrilateral. In example 5 we drew the Thales circle because there both opposite angles were 90°. Finally, we do a difficult example where it is essential to discover the cyclic quadrilaterals.

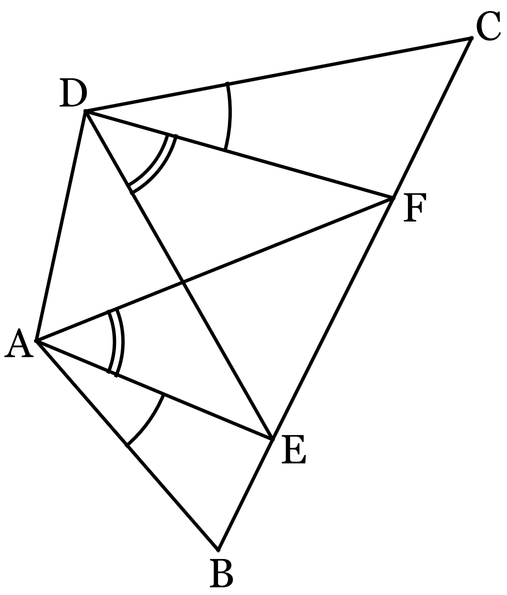


Figure 17: Example 6

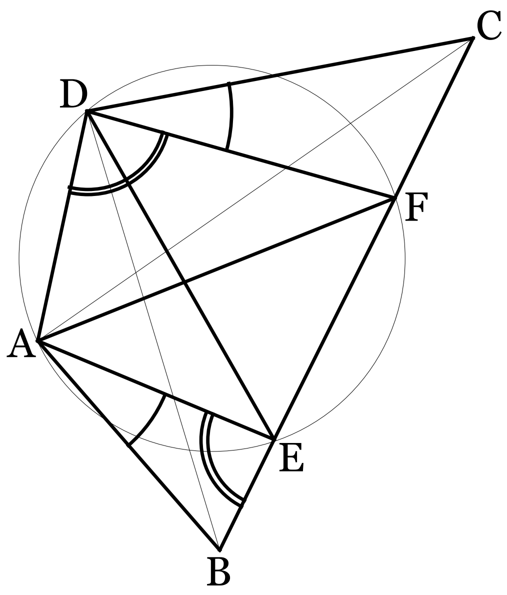


Figure 18: Solution to example 6

**Example 6** *The points $E$ and $F$ lie on the line $BC$ of the convex quadrilateral $ABCD$, where $E$ is closer to $B$ than $F$ (Fig. 17). Let $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$ hold.*
*Show $\angle FAC = \angle EDB$.*

*Solution.* (Fig. 18) While trying to solve such problems, you often get lost in countless auxiliary lines and angles. So it is worth taking the time to make a decent sketch. I often do it this way: I carefully draw the basic framework with a pen and draw in the rest with a pencil in different thicknesses depending on the importance. This has the advantage of being able to erase if needed without having to start all over again.
The first thing to notice in this example is that the two angles over the line $EF$ are equal: a situation like the Inscribed Angle Theorem! With its inversion we can conclude that $AEFD$ must be a cyclic quadrilateral. The fact that the inverse of the Inscribed Angle Theorem holds could be prove like Theorem 6 with Working Backward, but it is also common knowledge and therefore we skip the proof.
Now comes the difficult part of the problem. We have now found a cyclic quadrilateral and of course we still have to use the second prerequisite somehow. How can we combine

this cleverly? If one does not get any further, the following consideration is often helpful as an inspiration:
We assume that what we eventually want to show is correct (here $\angle FAC = \angle EDB$). What follows from this? Together with the premise and the inversion of the Inscribed Angle Theorem it follows that $ABCD$ is a cyclic quadrilateral. This is not yet a proof, of course, but we can now be sure that it must hold! We now try to prove this directly from the premises. Reversing the previous reasoning, we can then finish solving the problem.
We prove that $ABCD$ is a cyclic quadrilateral by showing that $\angle ADC + \angle ABC = 180°$ holds.

$$\angle ABC = 180° - \angle BEA - \angle BAE = \angle CEA - \angle BAE$$

For the next steps we first need the condition $\angle BAE = \angle CDF$, then that $AEFD$ is a chyclic quadrilateral

$$\angle ABC = \angle CEA - \angle BAE = \angle CEA - \angle CDF = 180° - \angle FDA - \angle CDF = 180° - \angle ADC$$

Therfore $ABCD$ is indeed a cyclic quadrilateral, from this follows quickly what we are looking for:

$$\angle FAC = \angle BAC - \angle BAF = \angle BDC - \angle BAF = \angle BDC - \angle EDC = \angle EDB$$

□

# 5 Tips for the exam

1. Be sure to take a compass and a ruler or set square with you. Often a geometric relationship that is important for the solution only becomes apparent when you have constructed a sketch exactly. It is also good to use different colors for the overview.
2. Don't be afraid to draw many different sketches. If you lose track, be persistent and try again.
3. Once you have found the proof from a sketch, you must write it down correctly, otherwise you may lose a lot of points. You must make it clear in your proof that you have seen all the steps necessary to solve it. This also applies to the tasks of the other topics. It is best to go through the proof again at the end and add to it if you think that something might be missing (it is better to write too much). In geometry, the following points in particular must not be forgotten:
   (a) If you have introduced placeholders (like $\alpha, \ldots$) you have to write at the beginning of the proof how they are defined (e.g. "'If $\alpha = \angle DAN$, $\beta = \ldots$"'). Exception: If a triangle $ABC$ is given then, if nothing else is defined, $\angle CAB = \alpha, \angle ABC = \beta, \angle BCA = \gamma$.
   (b) The same applies to auxiliary points, lines, circles, etc. used. As soon as an object not mentioned in the problem is used, it must be clear how it is defined.

(c) All the theorems mentioned in this script may be used without mentioning their name every time (because you need them all the time). But that doesn't mean that you don't have to write down every step. For example, if $ABCD$ is a cyclic quadrilateral, you could simply write "$\angle BCA = \angle BDA$ (arc $AB$)" and you wouldn't need to mention the Inscribed Angle Theorem.