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Induction Exercises

1 Exercises

Beginner

1.1 Prove that $n^2 + n$ is even for any natural number n.

1.2 Let $n \geq 3$. Prove that the sum of the internal angles of an n-gon is $(n-2) \cdot 180°$.

1.3 Show that for all $n \in \mathbb{N}$:

(a) $$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

(b) $$1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$$

(c) $$1^2 + 2^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(d) $$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n-1}{n!} = \frac{n! - 1}{n!}$$

(e) $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

1.4 Find all natural numbers n such that $3^n > n!$.

1.5 Prove that for all $n \in \mathbb{N}$:
$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

1.6 Prove that $7^{2n} - 2^n$ is divisible by 47 for all $n \in \mathbb{N}$.

Advanced

1.7 Show that all natural numbers can be written as a sum of powers of 2, all with different exponents.

1.8 Show that numbers $1007, 10017, 100117, 1001117, \ldots$ are all divisible by 53.

1.9 Let there be n lines cutting the place into regions (here, a region is a maximal convex area that doesn't intersect with a line). Prove there are at most $\frac{n^2+n+2}{2}$ regions.

1.10 (Preliminary round 2016, problem 2) Quirin has n blocks with heights ranging from 1 to n and would like to line them up, such that his cat can walk from the left end to the right end. The cat can only move from one block to another if the ending block is taller by 1 or shorter than the starting block. The cat starts at the left end. In how many ways can Quirin line up the blocks?

For example: $n = 5$, $3 - 4 - 5 - 1 - 2$ *is a possible arrangement but* $1 - 3 - 4 - 5 - 2$ *isn't.*

1.11 Let there be n points in the plane coloured in either blue or red. All red points are connected to a blue point by an edge. Prove there are at most $\frac{n^2+n-2}{4}$ edges.

1.12 How many subsets of $\{1, 2, \ldots, n\}$ have no pairs of consecutive elements ?

1.13 Let there be $n \geq 2$ people seated in a line in a restaurant. There are 3 dishes to choose from. No one wants to eat the same dish as a neighbour. In how many ways can the dishes be ordered so that everyone is satisfied?

1.14 Let there be n books et 3 platforms numbered from 1 to 3. At the beginning, each book is placed on platform 1, in decreasing order by size starting from the bottom. For each step, it is possible to move a book from one platform to another (or atop a pile on another platform), as long as a book is never placed above a smaller book. How many steps are necessary in order to transfer all books from platform 1 to platform 3 ?

Olympiad

1.15 Let $n \geq 6$. Prove it is possible to section a square into n smaller squares.

1.16 Let $n \in \mathbb{N}$. Consider a $2^n \times 2^n$ chessboard with one missing square. Prove a L-triomino paving exists.

1.17 There are n identical cars on a circular track (randomly spread out over the track). Their oil added up is just enough for one car to complete a full lap on the track. We pick a car to drive, the others remain at rest. When the moving car passes by a resting car, it takes the oil of the resting car. Prove that it is possible to choose a car such that it is possible to complete a full lap.

1.18 Let there be n candies in a jar. Alice and Bob play the following game: each turn, a player must eat a non-zero amount of candies less or equal to half the number of remaining candies. Alice takes the first turn. If there is only one candy after a player's turn, that player is the loser. For which n does Bob have a winning strategy ?