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Induction

One of the most important techniques in mathematics is *induction*. Let's start with an analogy. In order to go up a stepladder, first, one climbs up the first step. Then, from each step, one climbs up to the next step. As long as one is capable of

- reaching the first step,
- going from one step to the next,

it is possible to reach any step on the stepladder.

Would you like a different analogy? The 90's kids surely remember the famous Dutch television broadcast Domino Day. The idea was, year after year, to break the world record for falling dominoes in a chain. Enthusiasts would spend months lining up dominoes for the big day. What did the builders have to watch out for when lining up the dominoes? Well, first of all, you have to make sure that when a domino falls, it will cause the next one to fall. In addition, there must be a way to initiate the fall of the very first domino in the chain (often the work of a guest star). If these two conditions are satisfied, then all the dominoes will fall.

Let's move on to the mathetical formulation of the concept of induction. For all integers $n \geq 1$ let there be a statement $A(n)$. More precisely, $A(n)$ is a mathematical statement depending on n. The following are examples of statements:

1. $A(n)$: $1 + 2 + \ldots + n = n(n+1)/2$,
2. $B(n)$: *the number* n *is even,*
3. $C(n)$: *there exists at least one prime* p *such that* $n \leq p < 2n$.

Statements are either true or false. For example, $B(n)$ is true if and only if there exists an integer k such that $n = 2k$. The statement $A(n)$ is true for all $n \geq 1$ and statement $C(n)$ is true for all $n \geq 2$ (a famous number theory theorem known as *Bertrand's postulate*).

Suppose we want to prove that a certain statement $A(n)$ is true starting from a value n_0, i.e. for all $n \geq n_0$. How do we proceed? For example, we could begin by proving that $A(n_0)$ is true (climbing up the first step of the stepladder). Then, we could continue the proof by showing that is $A(n)$ is true, then $A(n+1)$ is definitely also true (it is possible to climb up to the next step from the previous step). Therefore, all statements $A(n)$ for $n \geq n_0$ are true (it is possible to reach any step of the stepladder).

The above method is called *proof by induction*. Synthetically, a proof by induction is made up of two steps that we summarise in the following theorem.

Theorem 1 (Classic induction) *Let* $A(n)$ *be a statement and* n_0 *an integer. Suppose*

1. *base case:* $A(n_0)$ *is true,*

2. *induction step* $A(n)$ *is true* $\Rightarrow A(n+1)$ *is true. Therefore statement* $A(n)$ *is true for all* $n \geq n_0$.

Let's look at a few examples.

Example 1 *Prove that for all natural numbers* $n \geq 1$

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Solution. In this case, the statement is the following:

$$A(n) : 1 + 2 + \ldots + n = \frac{n(n+1)}{2}, n \geq 1.$$

Our goal is to prove $A(n)$ is true for all $n \geq 1$ (which means $n_0 = 1$). Using induction:

1. **base case**, i.e. $A(1)$ is true:

   We obtain $A(1)$ as per the following:

   $$A(1) : 1 = \frac{1 \cdot (1+1)}{2}.$$

   $A(1)$ is therefore true.

2. **induction step**, i.e. $A(n) \Rightarrow A(n+1)$:

   Suppose $A(n)$ true. We want to prove that $A(n+1)$ is also true. We compute $A(n+1)$ to obtain:

   $$A(n+1) : 1 + 2 + \ldots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2}.$$

   We calculate the sum $1 + \ldots + (n+1)$. We obtain

   $$\begin{aligned}\underbrace{1 + 2 + \ldots + n}_{=\frac{n(n+1)}{2}} + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}.\end{aligned}$$

   In the first step, we used the formula that gives $A(n)$. We have proved that if $A(n)$ is true, then $A(n+1)$ is also true.

In conclusion, we have proved $A(1)$ and that if $A(n)$ true implies $A(n+1)$ also true. Therefore $A(n)$ is true for all $n \geq 1$. □

The following is an example of the applications of the induction method in combinatorics.

Example 2 (Two colour theorem) *Let there be* $n \geq 0$ *distinct lines in the plane. Prove that the regions delimited by these* n *lines can be coloured with at most two colours such that all pairs of bordering regions are of the same colour.*

Solution. Let $A(n)$ be the statement we want to prove. The base case, i.e. the case where the number of lines equals zero, is clearly true. A single colour can be used to colour the whole plane.

Let's now suppose that $A(n)$ is true and deduce $A(n+1)$ from it. Let there be $n+1$ lines in the place. Let d be a line among the $n+1$ lines. If we ignore d, then only n lines remain. As per the induction hypothesis, the areas delimited by these n lines can be coloured in black and white such that two neighbouring areas are never both black or both white.

Let's now add d back into the plane. Obviously, the current colouring is no longer valid. However, if we change the colouring of all regions on one side of d by inverting black and white regions, we obtain a valid colouring for the regions delimited by $n+1$ lines!

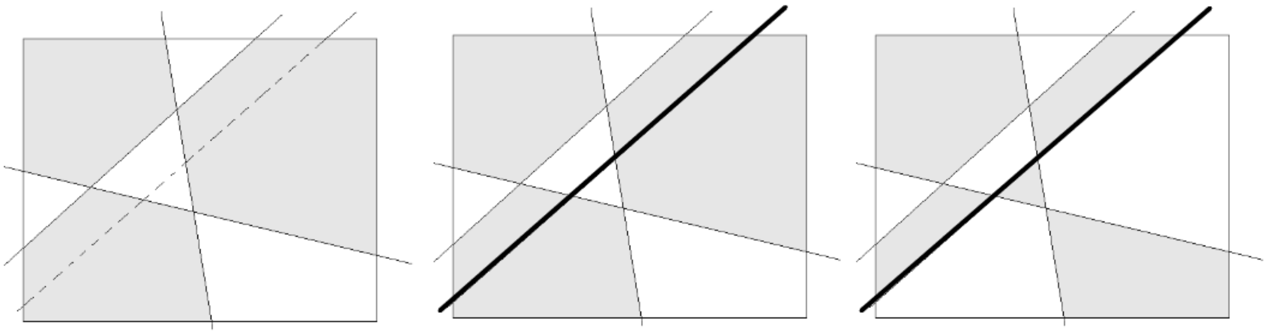

Figure 1: Illustration of the case with $n = 4$ lines delimiting 10 regions in the place. The line d (dashed, left image) is first ignored and the regions obtained with the $n - 1 = 3$ remaining lines are coloured with two colours. After adding back the line d (middle image) the colouring is no longer valid. By swapping the colours in the regions under d, we get a valid colouring (right image).

There are two cases to tackle.

1. Two regions that share a border that isn't part of d have different colours. In fact, the common border which isn't part of d is therefore part of one of the n other liens. These two regions were of different colours before the addition of d. Depending on the side of d they are in (since their border isn't part of d, they are on the same side), either the colours haven't been changed, or they have been swapped. Regardless, the two regions have different colours.
2. If the two regions had a shared border that is part of d, then the two regions were the same colour before the colour swap. Their colours are now different.

In conclusion, the colouring for the regions delimited by $n+1$ lines, based on the colouring for n lines, is valid. The induction step has been proven and the desired conclusion has been reached. $\square$

The following example illustrates the concept of *strong induction.* We want to prove that $A(n)$ is true for all $n \geq n_0$. Like in classical induction, we prove the base case that $A(n_0)$ is true. In strong induction, instead of proving that $A(n+1)$ is true with only the hypothesis that

$A(n)$ is true, the induction step consists of the proof that $A(n+1)$ is true by supposing that $A(k)$ is true for all $n_0 \leq k \leq n$ (not only $A(n)$). Hence the qualifier "strong"; we are using more hypotheses in order to reach the same conclusion. The following theorem sums up strong induction.

Theorem 2 (Strong induction) *Let $A(n)$ be a statement and n_0 be an integer. Suppose*

1. *base case: $A(n_0)$ is true,*
2. *induction step: for all $n \geq n_0$,*

$$A(k) \text{ is true for all } n_0 \leq k \leq n \Rightarrow A(n+1) \text{ is true.}$$

Therefore the statement $A(n)$ is true for all $n \geq n_0$.

Example 3 *Any natural number $n \geq 2$ has a prime decomposition (in other words, n can be written as a product of prime numbers).*

Note: we will not cover the uniqueness of the prime decomposition in this example.

Solution. We use strong induction on the statement.

$$A(n) : n \text{ can be written as a product of primes}, n \geq 2.$$

1. **base case**:

   Since 2 is a prime number, we can simple write $2 = 2$ (the product contains a single prime number).

2. **induction step**

   Suppose that for a certain integer $n \geq 2$, all integers $2 \leq k \leq n$ can be decomposed into prime factors. Our goal is to prove that $n+1$ also has a prime decomposition. If $n+1$ is prime, then the proof is finished, as we get $n+1 = n+1$ like in the base case. If $n+1$ is not prime, then $n+1$ can be written as the product of integers $a > 1$ and $b > 1$. Since a and b are less or equal to n (otherwise ab would be greater then $n+1$), a and b can be written as the product of primes as per the strong induction hypothesis. Therefore $n+1 = ab$ is also a product of primes and thus the induction step is true for all $n \geq 2$.

The base case and the induction step have been verified and we have reached our conclusion. □

Before we end, let's mention that there are types of induction other than classical induction and strong induction. For example, it is sometimes not easy to prove that $A(n) \Rightarrow A(n+1)$, while proving $A(n) \Rightarrow A(n+2)$ is. In order to prove that $A(n)$ is valid for all n, it is necessary to prove **two consecutive base cases** (eg. $A(1)$ et $A(2)$).

Another example used to prove inequalities is the following. Suppose we want to prove $A(n)$ for all $n \geq 1$. We could prove

1. $A(1)$ is true,
2. $A(n) \Rightarrow A(2n)$ for all $n \geq 1$,

3. $A(n) \Rightarrow A(n-1)$ for all $n \geq 2$.

The conclusion that $A(n)$ is true for all $n \geq 1$ is left as an exercise to the reader.