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Combinatorics Exercises

1 Counting tasks

Beginner

1.1 How many positive four-digit numbers are there, with:

a) all digits being the same? $\mathbf{9}$

b) an odd first digit? $\mathbf{5 \cdot 10^3}$

c) an even first digit? $\mathbf{4 \cdot 10^3}$

d) digits being different? $\mathbf{9 \cdot 9 \cdot 8 \cdot 7}$

e) no two adjacent digits being the same? $\mathbf{9^4}$

1.2 How many different ways are there to divide 10 people into two basketball teams with 5 people in each? $\boldsymbol{\binom{10}{5}/2}$

1.3 How many ways are there to make a bouquet of 12 flowers using three types of roses? $\boldsymbol{\binom{14}{12} = \binom{14}{2}}$

1.4 How many different permutations of the letters of the following words are there?

a) BIKES $\mathbf{5!}$

b) PAPERS $\boldsymbol{\frac{6!}{2!}}$

c) COFFEE $\boldsymbol{\frac{6!}{2! \cdot 2!}}$

d) MINIMUM $\boldsymbol{\frac{7!}{3! \cdot 2!}}$

1.5 Each of the four players A, B, C, D receives thirteen cards (from a deck of 52 cards). How many ways are there to distribute the cards? $\boldsymbol{\binom{52}{13} \cdot \binom{39}{13} \cdot \binom{26}{13}}$

Advanced

1.6 A group of 12 people wants to go on a boat trip and should be divided between three boats. The first boat fits 5 people, the second 4 and the last one 3. How many ways are there to divide these people between the boats? $\boldsymbol{\binom{12}{5} \cdot \binom{7}{4}}$
How many ways are left if we know that there is a married couple among these people that does not want to be separated? $\boldsymbol{\binom{10}{3} \cdot \binom{7}{4} + \binom{10}{5} \cdot \binom{5}{2} + \cdot \binom{10}{5} \cdot \binom{5}{4}}$

1.7 How many four-digit numbers are there, with:

a) exactly three different digits? $\boldsymbol{\binom{4}{2} \cdot 9 \cdot 9 \cdot 8}$

b) at least two same digits? $\mathbf{9 \cdot 10^3 - 9 \cdot 9 \cdot 8 \cdot 7}$

c) two even and one odd digits? $\mathbf{4 \cdot 3 \cdot 5^3 + \cdot 5 \cdot 3 \cdot 5^3}$

2 Other tasks

Beginner

2.1 Let $k \leq n$ be two natural numbers. How many ways are there to distribute k different balls to n children so that each child gets at most one ball? $\mathbf{\binom{n}{k}}$

2.2 We are given with 2 parallel lines. We choose 10 points on the first line and 11 points on the second. How many

a) quadrilaterals (polygons with 4 vertices) $\mathbf{\binom{10}{2} \cdot \binom{11}{2}}$

b) triangles $\mathbf{\binom{10}{2} \cdot 11 + \cdot \binom{11}{2} \cdot 10}$

with vertices in the chosen points are there?

2.3 How many positive integers smaller than 2014 are divisible by 3 or 4 but not by 5? $\mathbf{\lfloor \frac{2014}{3} \rfloor + \lfloor \frac{2014}{4} \rfloor - \lfloor \frac{2014}{12} \rfloor - \lfloor \frac{2014}{20} \rfloor - \lfloor \frac{2014}{15} \rfloor + \lfloor \frac{2014}{60} \rfloor}$

2.4 How many six-digit numbers are there so that the following holds:
Every next digit is strictly smaller than the previous one? $\mathbf{\binom{10}{6}}$

Advanced

2.5 n people are sitting at a round table. Two seating arrangements are considered same if each person has the same two neighbors. How many different seating arrangements are there? $\mathbf{\frac{n!}{2n}}$

2.6 How many ways are there to put 8 indistinguishable rooks on a chessboard so that no two rooks threaten each other? $\mathbf{8!}$

2.7 How many integer solutions of $x + y + z + w = 100$ are there if we know that $x, y, z, w \geq 8$? $\mathbf{\binom{71}{68}}$

2.8 A lotto ticket is a subset of $\{1, 2, \dots, 45\}$ with 6 elements. How many different tickets are there and how many of them contain two consecutive numbers? $\mathbf{\binom{45}{6} - \binom{40}{6}}$

2.9 A spider has one sock and one shoe for each of its eight legs. How many ways are there to put them on if it has to wear the sock first for each leg? $\mathbf{\binom{16}{2} \cdot \binom{14}{2} \cdot \binom{12}{2} \cdots \binom{4}{2} = \frac{16!}{2^8}}$

Olympiad

2.10 How many ways are there to choose two disjoint subsets from a set with n elements if we do not care about the order? (Note: empty set is also a subset.) $\mathbf{\frac{3^n+1}{2}}$

2.11 How many subsets with an even number of elements can be chosen from a set with n elements? $\mathbf{2^{n-1}}$

2.12 In some language there are n letters. A sequence of letters is called a word if and only if there are no two same letters between any two same letters.

a) How many letters can there be in a word at most? **3n**

b) How many words of maximum length are there? $\mathbf{n! \cdot 2^{n-1}}$

3 Tasks from previous olympiads

It makes sense to practice using the tasks from previous rounds. On one hand, they are on the same level in terms of difficulty as this year tasks, on the other hand, their solutions are on the homepage `www.imosuisse.ch`. However, you should always work on the tasks yourself first and only look at the sample solutions afterwards!

Preliminary round 2008, 2nd task A *way* in the plane starts in the point $(0,0)$ and ends in the point $(6,6)$. At every step one can go either 1 to the right or 1 up. How many paths are there that does not contain the point $(2,2)$ or the point $(4,4)$?

Preliminary round 2009, 2nd task Consider n children with different heights. How many ways are there to line up these children so that every child, except for the tallest one, has a neighbor taller than him?

Preliminary round 2010, 3rd task How many ways are there to assign one of the numbers $1,2,3,\dots,10$ to each corner of a cube so that each number is used at most once and that for each side the sum of the numbers in the four adjacent corners is odd?

Preliminary round 2011, 4th task We are given with a circular bus route with $n \geq 2$ stops (we can go in both directions). We call the distance between two adjacent stops a segment. One of the stops is Zurich. A bus should start in Zurich, travel via exactly $n+2$ segments and then get back in Zurich. The bus should visit each stop at least once. It can turn around at each stop. How many possible bus routes are there?

Preliminary round 2012, 2nd task We are given with $6n$ chips in $2n$ different colors, so that there are exactly 3 chips of each color. We should distribute the chips into two stacks A and B so that both stacks contain the same number of chips and no stack contains three chips of the same color. How many ways are there to do this if

a) the chips order within the stacks does not matter?

b) the order matters?

Preliminary round 2013, 3rd task We call a natural number sympathetic if its digits in the decimal system satisfy the following two conditions:

a) Every digit $0,1,\dots,9$ occurs at most once.

b) If A is even and B is odd, then there are exactly $\frac{A+B-1}{2}$ digits between A and B.

Determine the number of sympathetic numbers.

Preliminary round 2014, 3rd task How many eight-digit natural numbers are there for which each digit is either strictly greater or strictly less than all digits to the left?
Example: 45326791