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Combinatorics

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1 Divide and Conquer

An important component of combinatorics is counting. A typical question would be:

We have a hundred capuchin monkeys. How many ways are there to form a group of eighteen animals and to feed one of them a mealworm?

This chapter deals with various methods that allow us to answer questions like these.

The most important approach in combinatorics is **Divide and Conquer**:

1. Divide the problem into sub-problems.
2. Solve the sub-problems.
3. Using solutions of sub-problems, construct a solution for the original problem.

If you look closely, you will see that most of what we are going to do here (and not just that) follows this general recipe. So let us begin.

Product rule: If a selection process consists of r subprocesses, that are *independent* from each other, so that there are exactly n_k choices in the kth process, then the total amount of choices is equal to

$$n_1 \cdot n_2 \cdot \ldots \cdot n_r.$$

Most of the time we apply the product rule automatically.

Example 1

a) One can choose lasagna, pizza or gnocchi as a main course and Torta della Nonna or tiramisu as a desert. How many ways are there to order main course and desert?

b) How many natural numbers have exactly n digits when written in the decimal system?

c) A set consisting of n elements has 2^n subsets.

d) The natural number $n = p_1^{a_1} p_2^{a_2} \cdot \ldots \cdot p_r^{a_r}$ where p_i are different prime numbers, has exactly $(a_1+1)(a_2+1) \cdot \ldots \cdot (a_r+1)$ different positive dividers.

Solution.

a) There are three ways to choose the main course and two for the desert. That means there are 6 ways in total.

b) The first digit is not 0 because in this case the number would have less than n digits. So there are 9 ways to choose the first digit and 10 ways to do that for every other digit. As we choose digits independently from each other, we can apply the product rule. That means, there are $9 \cdot 10 \cdot \ldots \cdot 10 = 9 \cdot 10^{n-1}$ such numbers.

c) We can decide for each element independently whether it is a part of the subset. That means there are 2^n ways. (Note that empty set is also a subset)

d) A divider can be determined by the exponent b_k that corresponds to the prime p_k in prime factorization. Here $0 \le b_k \le a_k$, therefore there are $a_k + 1$ ways to choose b_k. (Note that 1 is also a divider)

□

To count something, one always determines the size of the set of the things one wants to count. Before going deeper into that, let us first introduce some useful definitions and notations:

If A is a finite set, we denote the *amount* of elements in A as $|A|$. The *union* of two sets A and B is denoted as $A \cup B$, it consists of all elements that are in A or in B. Note that the elements that are in A and B simultaneously are also a part of the union. The *intersection* of two sets is denoted as $A \cap B$, it consists of all elements that are in A and in B simultaneously. Two sets that have no common elements are called *disjunkt*. The *empty set* is denoted as $\varnothing$ or $\{\}$.

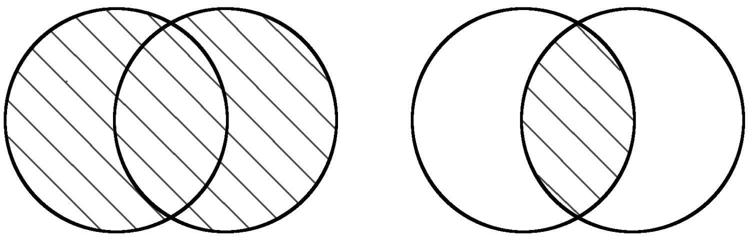


Figure 1: $A \cup B$ on the left, $A \cap B$ on the right

For $A = \{2, 4, 5\}$ and $B = \{1, 2, 5, 6\}$ we have $A \cup B = \{1, 2, 4, 5, 6\}$ and $A \cap B = \{2, 5\}$.

Sum rule: If $A = A_1 \cup A_2 \cup \ldots \cup A_n$ is a *disjoint* partition of the set A, then we have

$$|A| = |A_1| + |A_2| + \ldots + |A_n|.$$

We can see that Divide and Conquer appears in the sum rule too. If the elements of A cannot be directly counted, then it is divided into subsets, which it is easier to determine the size for. Afterwards the results are summed up. The sum product is natural for us and we use it subconsiously. For example, the total amount of people in the world is, of course, the amount of men and amount of women summed up. Later we will see interesting applications of this.

Now let us discuss a generalization of the sum rule for two special cases. The essential point of the sum rule is that the set A is divided into *disjoint* sets A_i. The issue is that often the partitions are not disjoint. Then we need the following formula:

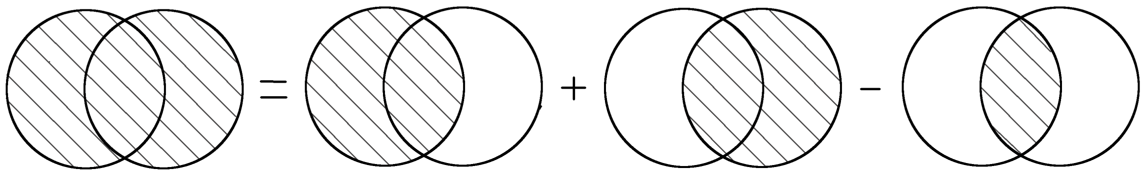
Inclusion-exclusion principle: It holds that

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

We see that in the first formula on the left we have the amount of elements that are in A or in B (or in both). We can also get this amount if we count elements of A and B and then subtract those that are present in both sets (because we count them twice). Consider the following picture.



In the second formula we get the amount of elements in $A \cup B \cup C$ in the same manner. We count all elements, then subtract those that are present in two sets and finally add those that are present in all three sets (because these elements are at first counted three times and then subtracted three times, therefore they should be added once). Draw an analogous diagram to the above picture to get a better understanding of the formula!

The inclusion-exclusion principle is useful because it is often much easier to count things that are in several sets at the same time than to count things that are at least in one of the sets. Consider a typical example:

Example 2 *How many ways are there to divide $n \geq 3$ candies among three children, so that every child gets at least one candy (we consider the candies to be distinct)?*

Solution. According to the product rule, there are 3^n ways to divide the candies if there are no additional constraints. We should subtract the ways when *at least* one kid gets nothing. We can count these ways with the inclusion-exclusion principle. Let us denote A, B and C the sets of ways when the first, second and third kids respectively gets nothing. We want to count $|A \cup B \cup C|$. According to the product rule, we have $|A| = 2^n$ and because of the symmetry $|B| = |C| = 2^n$. Apart from that, we have $|A \cap B| = 1$, because there is only one way to divide the candies so that the first and the second kids get nothing, namely when the third gets all the candies. Similarly, we get $|B \cap C| = |C \cap A| = 1$. Finally, $|A \cap B \cap C| = 0$ as at least one kid gets a candy. Therefore, we get

$$|A \cup B \cup C| = 3 \cdot 2^n - 3 \cdot 1 + 0,$$

and the amount of ways to divide the candies is then equal to $3^n - 3 \cdot 2^n + 3$. $\square$

2 The four fundamental selection processes

The four fundamental selection processes can be described with the contextualisation of **drawing k balls from an urn with n balls**. More precisely, one can imagine the urn with n distinguishable balls, which are numbered from 1 to n. Additionally, one should specify whether the ball is put back into the urn after drawing or not. Besides, one can consider the draw order important or not. According to the product rule, we distinguish between four cases.

1. **Without putting balls back, order matters.**
   Here we count the number of ways to draw k balls from an urn with n balls. Note that after drawing a ball, we do not put it back. Moreover, we care about the order in which we draw the balls. Examples:

   - Number of ways to choose k people out of n people and then form a queue out of them (number them from the front to the back).
   - Number of possible ways for the first k horses to finish in a horse race with n horses (note that here we assume no two horses finish at the same time).
   - Number of words of length k with all letters distinct from an alphabet with n letters.

   There are n ways to choose the first ball, $(n-1)$ for the second as we cannot choose the one we chose in the previous step. So for the third ball it is $(n-2)$ ways left, etc. Therefore, the total amount of ways is equal to

   $$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

   Here we have $n! = n(n-1) \cdot \ldots \cdot 2 \cdot 1$, which is a short notation for n *factorial*. Let us consider an important case $n = k$. This is the case where we just rearrange the things, which is called a *permutation*. There are exactly $n!$ permutations of a set with n elements.

2. **Without putting balls back, order does not matter.**
   Examples:

   - Number of ways to choose a team of k people from n people.
   - Number of subsets with k elements in a set with n elements.
   - Number of ways to draw k numbers from $\{1, 2, \ldots, n\}$ in a lottery.

   We denote that number as $\binom{n}{k}$ (we read this as n *choose* k) and call it *binomial coefficient*. What is this number equal to? At first we can consider the order when drawing k balls, which gives us $n!/(n-k)!$ ways. However, there are exactly $k!$ sequences that can be formed out of the drawn balls. Note that in this case we consider the sequences that only differ in the order as the same. That means we counted every possible draw $k!$ times, therefore we get

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

3. **With putting balls back, order matters.**
   Examples:

   - Number of words of length k from an alphabet with n letters.
   - Number of combinations for a combination lock.

   There are exactly n ways for each of the k balls as we put the balls back. Using the product rule, we get

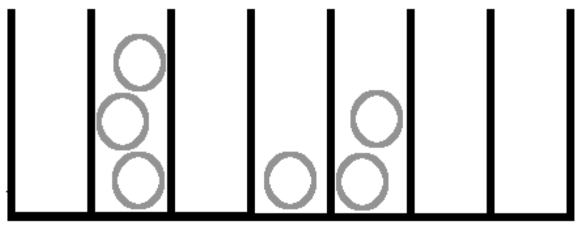
   $$n \cdot n \cdot \ldots \cdot n = n^k.$$

4. **With putting balls back, order does not matter.**
   Examples:

   - Number of ways to form a fruit bowl with k fruits if there are n different types of fruits available.
   - Number of possible outcomes when rolling k dices. (Note that these outcomes are not equally likely.)

   Two draws are identical if for every number we drew the same amount of balls with this number (remember that all balls have numbers from 1 to n). Therefore, we can specify such draw if for every number we indicate how many times the ball with this number was drawn. Similarly, we could divide k balls between n boxes.

   You can see a possible draw for $k = 6$, $n = 7$ in the following picture. Here we drew a 2 three times, a 4 once and a 5 twice.

   This picture is not so useful to actually get the answer, consider another visualization:

   $$|\,\mathrm{OOO}\,||\,\mathrm{O}\,|\,\mathrm{OO}\,||$$

   In this picture, box borders are represented as vertical lines and balls as 0s. We leave out the first and the last box border as they are always at the very beginning or in the very end respectively. Every line now corresponds to the increasing the ball's number by 1. Again, we can see that we drew a 2 three times, a 4 once, and a 5 twice. The remaining spaces have no balls between them, which means that we did not draw any balls with numbers 1, 3, and 6.

After we encoded a draw as described above, it is much easier to count the number of ways to draw the balls. Obviously, it is possible to reconstruct a draw from balls and vertical lines. Furthermore, there is exactly one such sequence for every draw. To specify a sequence, it is enough to choose where to put vertical lines (or balls, which gives the same result). Note that we choose from $k + (n-1)$ places as there are $n-1$ vertical lines and k balls. Therefore, the number we were looking for is

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}.$$

However, these four types rarely come in their purest form. We usually need to combine these methods cleverly to get the final result. To solve a concrete problem, one often consciously or unconsciously enumerates things, so it is easier to count them. However, it is important to divide by the appropriate factor in the end to remove results we counted more that once because of enumerating. At first one might make mistakes here, so the only way to get rid of that is practice.

Example 3 *There are $2n$ tennis players competing in a tournament. In the first round, all participants play exactly once and all games are one on one. How many ways are there to assign players to n games in the first round?*

Solution. We give two solutions, so you can practise more. Let A_n be the number we are looking for.

1st solution
Form a queue from $2n$ players, there are $(2n)!$ ways to do that. Then form pairs $(1,2), (3,4), \ldots, (2n-1, 2n)$; there is only one way to do this. The thing is that it only matters who plays against whom, but not who stands where. There are 2^n ways to permute the players with their opponent and $n!$ ways to permute the pairs. That means we need to divide by $2^n n!$ to eliminate ways we counted multiple times.

$$A_n = \frac{(2n)!}{2^n n!} = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

2nd solution
We can choose the first pair in $\binom{2n}{2}$ ways. There are $\binom{2n-2}{2}$ ways to choose the second pair and for the kth pair we have $\binom{2n-2(k-1)}{2}$ ways. However, we have enumerated n pairs, so we should divide by $n!$, so we get

$$A_n = \frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2} = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

□

Example 4 *How many ways are there to put 5 blue and 7 red books next to each other on a shelf, so that there are no two blue books next to each other?*

Solution. We give two solutions here too.

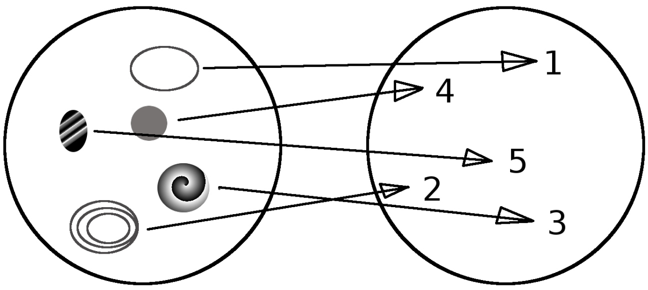
1st solution
We first distribute all blue books on the shelf. Now we need to put at least one red book into each of the four resulting spaces between the blue books. Afterwards, we can put the three remaining red books in any of the spaces, including the beginning and the end. We can do that in $\binom{8}{3} = 56$ ways.

2nd solution
We first distribute all red books on the shelf. Now we need to distribute the blue books between the beginning, the end and the resulting spaces between the red books. Furthermore, there can be at most one blue book in each of the previously mentioned spaces. As we cannot distinguish between the blue books, there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4/5!$ to do that. $\square$

The next chapter discusses a very important and common method in combinatorics. It comes into play if it is not possible to determine the size of the set with previously described methods.

3 Bijections

A mapping $f\colon A \to B$ between two sets A and B is called *bijective* or 1:1-*function* or *bijection* if there is a unique way to attain every element in B. The name comes from the fact that a bijective function establishes a 1:1 relationship between elements in A and elements in B, because each element in B has exactly one element in its preimage.



If there exists a bijective function $f\colon A \to B$ between some finite sets A and B, then we automatically get $|A| = |B|$. This is often very convenient. If you have a set A whose size you cannot determine, then you can try to find a bijection to another set with elements that are easier to count.

This is exactly what we did in the 4th case of the urn problem described above. We bijectively mapped the set of possible draws to the set of all sequences from k balls and $n - 1$ vertical lines. It is much easier to count these sequences. Let us consider more examples now.

Example 5 *Consider a convex polygon with n vertices. It holds that no three diagonals intersect in an interior point (a point that is not part of the edges). Determine the number of interior points of the polygon that are intersections of two diagonals.*

Solution. Each intersection point we are looking for lies on exactly two diagonals, since we know that no three diagonals intersect in an interior point. Consider two diagonals and the 4 vertices of the polygon in which they end. We can use these vertices to represent the intersection. One can easily see that there is exactly one way to choose two diagonals that intersect inside the polygon and whose endpoints are exactly these 4 vertices. This way we constructed a bijection from the set of all interior intersections of diagonals to the set of all subsets from 4 vertices. There are exactly $\binom{n}{4}$ ways to choose 4 vertices, so there are as many intersection points. $\square$

Example 6 (Number Theory I is required) *Let $M = \{0, 1, 2, \ldots, n\}$. A pair (a, b) of two numbers from M is called good if $n \mid a + 2b$ holds. Show that there are as many good pairs (a, b) with $a > b$ as with $b > a$.*

Solution. To solve the problem, we could try to count the number of good pairs with $a > b$ and $b > a$. But why should we? The exact number is not required! We can solve the problem much faster with bijection. The key observation here is that (a, b) is a good pair if and only if $(n - a, n - b)$ is also one. This follows directly from the equation $(a+2b)+((n-a)+2(n-b)) = 3n$. Moreover, we have $a > b \Leftrightarrow n-a < n-b$. Therefore, the mapping $(a, b) \mapsto (n - a, n - b)$ is a bijection from the set of good pairs with $a > b$ to the set of those with $b > a$. $\square$

Example 7 *How many binary sequences of length n that contain exactly m 01 blocks are there? (A binary sequence is a sequence of zeros and ones).*

Solution. We are looking for the sequences with exactly m $0 - 1$ changes. How can we count the number of $1 - 0$ changes? Obviously, there is always exactly one $1 - 0$ change between two $0 - 1$ changes, but whether it is at the beginning or in the end depends on the sequence. However, we can always force this by prepending (adding at the very beginning) 1 to the sequence and appending (adding in the very end) 0. After this modification, the new sequence of length $n + 2$ has exactly $m + 1$ such $1 - 0$ changes. Obviously, this modified sequence is uniquely determined by the positions of the $2m + 1$ changes from 0 to 1 or from 1 to 0. These always happen between some adjacent elements, i.e. in some of the $n+1$ spaces. There are exactly $\binom{n+1}{2m+1}$ ways to choose these positions. This is exactly what we were supposed to find. (Question: where is a bijection in this proof?) $\square$