Combinatorial Games - Exercises

1 States

1. A monkey is at the centre of a circular lake. There is a lion moving on the boundary of the lake who wants to catch the monkey. Even though the lion is 4 times as fast as the swimming monkey, prove that the monkey can escape the lake without getting caught and then run away with its superior speed on land.
2. (Lithuania 2010) In an $m \times n$ rectangular chessboard there is a stone in the lower leftmost square. Jakob and Nina move the stone alternately, starting with Jakob. In each step one can move the stone upward or rightward any number of squares. The player who moves it into the upper rightmost square wins. Find all (m, n) such that Jakob has a winning strategy.
3. (Saint-Petersburg 2001) The number 1,000,000 is written on a board. Kosta and Amir take turns, Kosta starting, consisting of replacing the number n on the board with $n-1$ or $\lfloor \frac{n+1}{2} \rfloor$. The player who writes the number 1 wins. Who has the winning strategy?
4. (Russia 2011) There are $N > n^2$ stones on a table. Viktor and Anna play a game. Viktor begins, and then they alternate. In each turn a player can remove k stones, where k is a positive integer that is either less than n or a multiple of n. The player who takes the last stone wins. Prove that Viktor has a winning strategy.
5. (IMO Shortlist 2004) David and Domen take turns writing a number as follows. Let N be a fixed positive integer. First David writes the number 1, and then Domen writes 2. Hereafter, in each move, if the current number is k, then the player whose turn it is can either write $k+1$ or $2k$, but no player can write a number larger than N. The player who writes N wins. For each N, determine who has a winning strategy.
6. (Russia 1999) There are 2000 devices in a circuit, every two of which were initially joined by a wire. The hooligans Schlaudraff and Ciolpan cut the wires one after another. Schlaudraff, who starts, cuts one wire on his turn, while Ciolpan cuts two or three. A device is said to be disconnected if all wires incident to it have been cut. The player who makes some device disconnected loses. Who has a winning strategy?
7. (Rioplatense Olympiad 2010) Clara and Marina play the following game. To start, Clara arranges the numbers $1, 2, \ldots, n$ in some order in a row and then Marina chooses one of the numbers and places a pebble on it. A player's turn consists of picking up and placing the pebble on an adjacent number under the restriction that the pebble can be placed on the number k at most k times. The two players alternate taking turns beginning with Clara. The first player who cannot make a move loses. For each positive integer n, determine who has a winning strategy.
8. (India 2013) A marker is placed at the origin of an integer lattice. Jerome and Gavin play the following game. Jerome starts the game and each of them takes turns alternatively. At each turn, one can choose two (not necessarily distinct) integers a and b, neither of which was chosen earlier by any player and move the marker by a units in the horizontal direction and b units in

the vertical direction. Gavin wins if the marker is back at the origin any time after the first turn. Determine whether Jerome can prevent Gavin from winning.

9. (Based on South Korea 2009). Consider an $m \times (m+1)$ grid of points, where each point is joined by a line segment to is immediate neighbours (points immediately to the left, right, above or below). A stone is initially placed on one of the points in the bottom row. Beth and Matthias alternately move the stone along line segments, according to the rule that no line segment may be used more than once. The player unable to make a legal move loses. Determine which player has a winning strategy.

10. There is one pile of N counters. Geertje and Giambattista play a game, moving alternately as follows. In the first turn of the game, Geertje may remove any positive number of counters, but not the whole pile. Thereafter, each player may remove at most twice the number of counters his opponent took on the previous move. The player who removes the last counter wins. Who has the winning strategy?

2 Symmetry

11. The positive integers from 1 to n are written on the blackboard. Yannis and Charles-Edouard are taking turns, where each player has to choose a number on the board and erase it as well as all its divisors. The player who erases the last number loses. Decide who has a winning strategy.

12. (Échecs Marseillais) The game of double chess is played like regular chess, except each player makes two moves in their turn (white plays twice, then black plays twice, and so on). Show that white can always win or draw.

13. Lovro and Jaka play a game on an $n \times m$ chessboard. Lovro starts by removing a corner square and then the two start taking turns, each deleting a square sharing an edge with the square that was deleted on the previous turn. The first player to not have a move loses. Depending on n and m, determine who has a winning strategy.

14. Miroslav and Anastasia each get an unlimited supply of identical circular coins. They take turns placing the coins on a finite square table, in such a way that no two coins overlap and each coin is completely on the table (that is, it doesn't stick out). The person who cannot legally place a coin loses. Assuming at least one coin can fit on the table and that Miroslav starts, prove that he has a winning strategy.

15. (Saint-Petersburg 1997) The number N is the product of k different primes ($k \geq 3$). Vicky and Glenn take turns writing composite divisors of N on a board, according to the following rules. One may not write N. Also, there may never appear two coprime numbers or two numbers, one of which divides the other. The first player unable to move loses. If Vicky starts, who has the winning strategy?

16. (USAMO 2004) Kaloyan and Dima play a game on a 6×6 grid. On their turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Kaloyan goes first and then the players alternate. When all of the squares have numbers written in them, in each row, the square with the greatest number in that row is coloured black. Kaloyan wins if he can then draw a path from the top of the grid to the bottom of the grid that stays in black squares, and Dima wins if he can't. (A path is a sequence of squares such that any two

consecutive squares in the path share a vertex). Find, with proof, a winning strategy for one of the players.

17. On a 5×5 board, Artur and Oleg take turns marking squares. Artur always writes an X in a square and Oleg always writes O. No square can be marked twice. Artur wins if he can make one full row, column or diagonal contain only Xs. Can Oleg prevent A from winning?

18. (IMO Shortlist 1994) Geoff and Dom play alternately on a 5×5 board. Geoff always enters a 1 into an empty square, and Dom always enters a 0 into an empty square. When the board is full, the sum of the numbers in each of the nine 3×3 squares is calculated and Geoff's score S is the largest such sum. What is the largest score Geoff can make, regardless of the responses of Dom?

19. (Italy 2009) Po-Shen and Alex play the following game. First Po-Shen writes a permutation of the numbers from 1 to n, where n is a fixed positive integer greater than 1. In each player's turn, he or she must write a sequence of numbers that has not been written yet such that either:

    - The sequence is a permutation of the sequence the previous player wrote, or
    - The sequence is obtained by deleting one number from the previous player's sequence.

    For example, if Po-Shen first writes 4123, Alex could write 3124 or 413. The player who cannot write down a sequence loses. Determine who has the winning strategy.

20. (USAMO 1999) The *Y2K Game* is played on a 1×2000 grid as follows. Boschini and Christian in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

21. (IMO Shortlist 2009) Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Massimiliano and Navneel, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

    a) Does the game necessarily end?

    b) Does there exist a winning strategy for the starting player?

22. (Russia 1999) There are three empty jugs on a table. Evan, Pranav, and Yufei put walnuts in the jugs one by one. They play successively, with the order chosen by Pranav in the beginning. Thereby Evan plays either in the first or second jug, Pranav in the second or third, and Yufei in the first or third. The player after whose move there are exactly 1999 walnuts in some jug loses. Show that Evan and Yufei can cooperate so as to make Pranav lose.

23. (Bulgaria 2005) For positive integers t, a, b, a (t, a, b)-game is a game played by Maxime and Titouan defined by the following rules. Initially, the number t is written on a blackboard. In his first move, the Maxime replaces t with either $t - a$ or $t - b$. Then, Titouan subtracts either a or b from this number, and writes the result on the blackboard, erasing the old number. After this, Maxime once again erases either a or b from the number written on the blackboard, and so on. The player who first reaches a negative number loses the game. Prove that there exist infinitely many values of t for which Maxime has a winning strategy for all pairs (a, b) with $(a + b) = 2005$.