MATHEMATICAL.
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OLIMPIADI DELLA MATEMATICA

IMO Selection 2024

First Exam

Bern

May 4, 2024

Duration: 4.5 hours

Difficulty: The problems are ordered by difficulty.

Points: Each problem is worth 7 points.

1. Let $n > 1$ be an odd integer with smallest prime divisor p. Assuming that any prime divisor q of n also divides n/q, prove that
$$\sqrt{n^{p+1}} \mid 2^{n!} - 1.$$

2. Let ABC be a triangle with circumcircle Γ. Let $D \neq A$ be the second intersection of the internal bisector of $\angle BAC$ with Γ. We define E to be the intersection of line CD with the line perpendicular to BC through B, and ω to be the circumcircle of ADE. The line parallel to AD passing through E intersects ω at $F \neq E$. Moreover, let the tangents to Γ at A and C intersect at T. Prove that TF is tangent to ω.

3. Determine all monic polynomials P with integer coefficients such that for all integers a and b, there exists an integer c such that $P(a)P(b) = P(c)$.

Good luck!

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Second Exam

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Bern
May 5, 2024

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

4. Let $a_1, \ldots, a_{2^{2024}}$ be a sequence of pairwise distinct positive integers. Define

$$S_n = \frac{1}{1+a_1} + \frac{a_1}{(1+a_1)(1+a_2)} + \cdots + \frac{a_1 a_2 \cdots a_{n-1}}{(1+a_1)(1+a_2)\cdots(1+a_n)}.$$

Determine how many sequences $a_1, \ldots, a_{2^{2024}}$ exist, such that $S_{2^i} = \frac{2^i}{2^i+1}$ for all $0 \leq i \leq 2024$.

5. Let $n \geq 4$ be an integer and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sequences of positive integers such that the $n+1$ products

$$\begin{gathered} a_1 a_2 \cdots a_{n-1} a_n, \\ b_1 a_2 \cdots a_{n-1} a_n, \\ b_1 b_2 \cdots a_{n-1} a_n, \\ \vdots \\ b_1 b_2 \cdots b_{n-1} a_n, \\ b_1 b_2 \cdots b_{n-1} b_n, \end{gathered}$$

taken in this order, form a strictly increasing arithmetic progression. Determine the smallest possible common difference of this arithmetic progression in terms of n.

Remark: An arithmetic progression is a sequence of the form $a, a+r, a+2r, \ldots, a+kr$ where a, r and k are integers and r is called the common difference.

6. Let $n \geq 2$ be an integer. Kaloyan has a $1 \times n^2$ strip of unit squares, where the i-th square is labelled with i for all $1 \leq i \leq n^2$. He cuts the strip into several pieces, each piece consisting of a number of consecutive unit squares. He then places the pieces, without rotation or reflection, on an $n \times n$ square such that the square is covered entirely and the unit square in the i-th row and j-th column contains a number congruent to $i+j$ modulo n.

Determine the smallest number of pieces for which this is possible.

Good luck!

IMO Selection 2024

Third Exam

Bern

May 18, 2024

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

7. Let $m, n \geq 2$ be integers. On each unit square of a $m \times n$ grid there is a coin. Initially all coins show heads. Jérôme repeatedly performs the following operation. First, he selects a 2×2 square within the grid and then does one of:

- Flipping all coins in the chosen 2×2 square except the top-right one.
- Flipping all coins in the chosen 2×2 square except the bottom-left one.

Determine all pairs (m, n) for which, at some point, Jérôme can make all coins show tails at the same time.

8. Determine all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$x\left(f(x) + f(y)\right) \geq f(y)\left(f(f(x)) + y\right)$$

for all $x, y \in \mathbb{R}_{>0}$.

9. Let ABC be an acute triangle with orthocenter H, satisfying $AC > AB > BC$. The perpendicular bisectors of AC and AB intersect line BC at R and S respectively. Let P and Q be points on lines AC and AB respectively, both different from A, such that $AB = BP$ and $AC = CQ$. Prove that the distances from point H to lines SP and RQ are equal.

Good luck!

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Fourth Exam

Bern

May 19, 2024

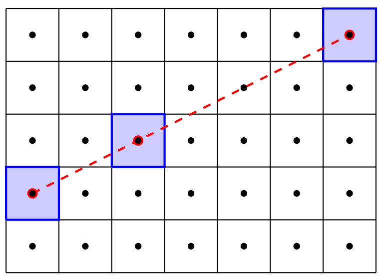
Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

10. Let ABC be a triangle with $AC > BC$. Let ω be the circumcircle of triangle ABC and let r be the radius of ω. Let point P lie on the segment AC such that $BC = CP$ and let S be the foot of the perpendicular from P to line AB. Let the line BP intersect ω again at $D \neq B$. Let Q lie on line SP such that $PQ = r$ and such that S, P and Q lie on the line in that order. Finally, let the line perpendicular to CQ from A intersect the line perpendicular to DQ from B at E.

Prove that E lies on ω.

11. Let $m, n \geq 3$ be integers. Nemo is given an $m \times n$ grid of unit squares with one chip on every unit square initially. They can repeatedly carry out the following operation: first, they pick any three distinct collinear unit squares and then they move one chip from each of the outer two squares onto the middle square. They may only do this operation if the outer two squares are not empty, but the middle square is allowed to be empty.

As a function of (m, n), either determine the maximum number of operations Nemo can make before they cannot continue anymore, or prove that they can carry out an arbitrarily large number of operations.



An example of three squares with collinear centres

12. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$\underbrace{f(f(\cdots f}_{bf(a)}(a+1)\cdots)) = (a+1)f(b)$$

holds for all $a, b \in \mathbb{N}$.

Good luck!