MATHEMATICAL.
OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

IMO Selection 2023

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Bern
May 13, 2023
First Exam

1. In a garden, there are 2023 rose bushes planted in a row. Each bush contains either red or blue roses. Vicky is taking a walk and wants to pick some of the flowers. She starts at a bush of her choice, and picks a rose from it to add to her basket. She then continues walking down the row and picks a single flower from each bush she visits. Vicky can skip some bushes, but she cannot skip two adjacent bushes. She can leave the garden at any point. Let r and b be the number of red and blue roses she picked, respectively. Determine the maximal value of $|r-b|$ Vicky can achieve, irrespective of the configuration of bushes.

2. Let S be a non-empty set of positive integers such that for any $n \in S$, all positive divisors of $2^n + 1$ are also in S. Prove that S contains an integer of the form

$$(p_1 p_2 \dots p_{2023})^{2023},$$

where $p_1, p_2, \dots, p_{2023}$ are distinct prime numbers, all greater than 2023.

3. Let ABC be a triangle, and let l_1 and l_2 be two parallel lines. For $i = 1, 2$, assume l_i meets the lines BC, CA and AB at X_i, Y_i and Z_i respectively. Suppose that the line through X_i perpendicular to BC, the line through Y_i perpendicular to CA, and finally the line through Z_i perpendicular to AB, determine a non-degenerate triangle Δ_i. Show that the circumcircles of Δ_1 and Δ_2 are tangent to each other.

Good luck!

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Bern
May 14, 2023
Second Exam

4. Let ABC and AMN be two similar, non-overlapping triangles with the same orientation, such that $AB = AC$ and $AM = AN$. Let O be the circumcentre of the triangle MAB. Prove that the points O, C, N and A lie on a circle if and only if the triangle ABC is equilateral.

5. The Tokyo Metro system is one of the most efficient in the world. There is some odd positive integer k such that each metro line passes through exactly k stations, and each station is serviced by exactly k metro lines. One can get from any station to any other station using only one metro line - but this connection is unique. Furthermore, any two metro lines must share exactly one station. David is planning an excursion for the IMO team, and wants to visit a set S of k stations. He remarks that no three of the stations in S are on a common metro line. Show that there is some station not in S, which is connected to every station in S by a different metro line.

6. Determine all positive integers $n \geq 2$ for which there exist n distinct real numbers $a_1, a_2, \ldots, a_n$ and a real number $r > 0$ such that

$$\left\{a_j - a_i \mid 1 \leq i < j \leq n\right\} = \left\{r, r^2, \ldots, r^{\binom{n}{2}}\right\}.$$

Good luck!

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Bern
May 27, 2023
Third Exam

7. Determine all monic polynomials $P(x) = x^{2023} + a_{2022}x^{2022} + \cdots + a_1 x + a_0$ with real coefficients such that $a_{2022} = 0$, $P(1) = 1$, and all roots of P are real and less than 1.

8. Let ABC be an acute triangle with $AC > AB$, let O be its circumcentre, and let D be a point on the segment BC. The line through D perpendicular to BC intersects the lines AO, AC and AB at W, X and Y, respectively. The circumcircles of triangles AXY and ABC intersect again at $Z \neq A$. Prove that if $OW = OD$, then the line DZ is tangent to the circle AXY.

9. Let G be a graph whose vertices are the integers. Assume that any two integers are connected by a finite path in G. For two integers x and y, we denote by $d(x, y)$ the length of the shortest path from x to y, where the length of a path is the number of edges in it. Assume that $d(x, y) \mid x - y$ for all $x, y \in \mathbb{Z}$ and define $S(G) = \{d(x, y) \mid x, y \in \mathbb{Z}\}$. Find all possible sets $S(G)$.

Good luck!

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Bern
May 28, 2023
Fourth Exam

10. Let $a, d > 1$ be two coprime integers. Define the sequence $(x_i)_{i\in\mathbb{N}}$ by setting $x_1 = 1$ and

$$x_{k+1} = \begin{cases} x_k/a & \text{if } a \text{ divides } x_k \\ x_k + d & \text{otherwise} \end{cases}$$

for all $k \geq 1$. Determine the largest non-negative integer n such that a^n divides at least one term of the sequence, or prove that no such n exists.

11. Denote by $\mathcal{F}$ the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the equation

$$f(x + f(y)) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Determine all rational numbers q such that for every function $f \in \mathcal{F}$, there exists some $z \in \mathbb{R}$ with $f(z) = qz$.

12. For a positive integer m, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$. Let n be a positive integer and let $\mathcal{S}$ be a non-empty collection of subsets of $[n]$. A function $f\colon [n] \to [n+1]$ is called *kawaii* if there exists $A \in \mathcal{S}$ such that for all $B \in \mathcal{S}$ with $A \neq B$ we have

$$\sum_{a\in A} f(a) > \sum_{b\in B} f(b).$$

Prove that there are always at least n^n *kawaii* functions, irrespective of $\mathcal{S}$.

Good luck!