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OLIMPIADI DELLA MATEMATICA

IMO-Selection 2022

First exam

7 May 2022

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

1. Let n be a positive integer. Prove that there exists a finite sequence S consisting of only zeros and ones, satisfying the following property: For any positive integer $d \geq 2$, when S is interpreted as a number in base d, the resulting number is non-zero and divisible by n.

Remark: The sequence $S = s_k s_{k-1} \cdots s_1 s_0$ *interpreted in base* d *is the number* $\sum_{i=0}^{k} s_i d^i$.

2. Let $ABCD$ be a convex quadrilateral such that the circle with diameter AB is tangent to the line CD, and the circle with diameter CD is tangent to the line AB. Prove that the two intersection points of these circles and the point $AC \cap BD$ are collinear.

3. A hunter and a rabbit are playing the following game on the cells of an infinite square grid. First, the hunter fixes a colouring of the cells using finitely many colours. After that, the rabbit secretly chooses a cell to start in. Each turn, the rabbit reports the colour of its current cell to the hunter and then secretly moves to an adjacent cell (sharing an edge) that has not been visited before.

The hunter wins if at any point of the game either:

- They can determine with certainty the current cell the rabbit is in.
- The rabbit cannot make a legal move.

Determine if the hunter has a winning strategy.

Good luck!

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Second exam
8 May 2022

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

4. Given a (simple) graph G with $n \geq 2$ vertices $v_1, v_2, \ldots, v_n$ and $m \geq 1$ edges, Joël and Robert play the following game with m coins:

i) Joël first assigns to each vertex v_i a non-negative integer w_i such that $w_1 + \cdots + w_n = m$.

ii) Robert then chooses a (possibly empty) subset of edges, and for each edge chosen he places a coin on exactly one of its two endpoints, and then removes that edge from the graph. When he is done, the amount of coins on each vertex v_i should not be greater than w_i.

iii) Joël then does the same for all the remaining edges.

iv) Joël wins if the number of coins on each vertex v_i is equal to w_i.

Determine all graphs G for which Joël has a winning strategy.

5. Let a, b, c, λ be positive real numbers with $\lambda \geq 1/4$. Show that

$$\frac{a}{\sqrt{b^2 + \lambda bc + c^2}} + \frac{b}{\sqrt{c^2 + \lambda ca + a^2}} + \frac{c}{\sqrt{a^2 + \lambda ab + b^2}} \geq \frac{3}{\sqrt{\lambda + 2}}.$$

6. Let $n \geq 2$ be an integer. Prove that if

$$\frac{n^2 + 4^n + 7^n}{n}$$

is an integer, then it is divisible by 11.

Good luck!

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Third exam
28 May 2022

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

7. Let n be a positive integer. Find all polynomials P with real coefficients such that

$$P(x^2 + x - n^2) = P(x)^2 + P(x)$$

for all real numbers x.

8. Johann and Nicole are playing a game on the coordinate plane. First, Johann draws any polygon $\mathcal{S}$ and then Nicole can shift $\mathcal{S}$ to wherever she wants. Johann wins if there exists a point with coordinates (x, y) in the interior $\mathcal{S}$, where x and y are coprime integers. Otherwise, Nicole wins. Determine who has a winning strategy.

9. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that TE is parallel to CD and TF is parallel to AD. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Good luck!

IMO-Selection 2022

Fourth exam
29 May 2022

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

10. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR intersect at one point.

11. Let $n \geq 2$ be an integer. Each of the squares of an $n \times n$ board contains a bit-coin with 0 on one side, 1 on the other. Initially, all bit-coins in the leftmost column show a 0. A move consists of one of the following:

- Within any row, look at the rightmost two neighbouring bit-coins that display different numbers (if they exist) and flip both of these as well as all bit-coins to their right.
- Within any column, look at the topmost two neighbouring bit-coins that display different numbers (if they exist) and flip both of these as well as all bit-coins above them.

Find the minimal value of k such that there always exists a sequence of moves resulting in at most k bit-coins showing 1.

12. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$x + f(yf(x) + 1) = xf(x + y) + yf(yf(x))$$

for all positive real numbers x and y.

Good luck!