MATHEMATICAL.
OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

IMO-Selection 2021

First exam

8 May 2021

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

1. Let ABC be a triangle where $\angle BAC = 90°$, with circumcenter O and incenter I. The angle bisector of $\angle BAC$ intersects the circumcircle of ABC in A and P. Let Q be the projection of P onto AB, and R be the projection of I onto PQ. Prove that RO bisects CI.

2. For each prime p, somewhere in the multiverse there exists a kingdom consisting of p islands numbered from 1 to p with a bridge between any pair of them. When Jana visits a kingdom, coronavirus restrictions mean she must obey the following rule: Directly after visiting island m, she can only cross over to island n if

$$p \mid (m^2 - n + 1)(n^2 - m + 1).$$

Show that there are infinitely many kingdoms such that Jana cannot travel to every island in this manner.

3. Let p be an odd prime. Arnaud has hung up $N \geq 1$ towels to dry on a washing line, each coloured either purple or yellow. Then, for each $1 \leq n \leq N$, he calculates what fraction of the first n towels are yellow, and writes down these N fractions in their irreducible form on a piece of paper. Julia finds the piece of paper the next day and notices that all of the fractions $\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$ are on the paper. Prove that

$$N \geq \frac{p^3 - p}{4}.$$

Good Luck!

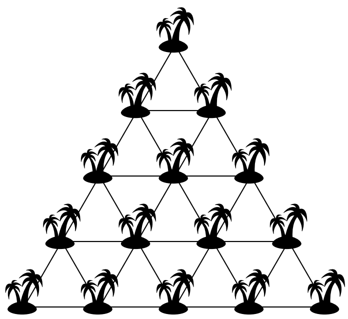
IMO-Selection 2021

Second exam
9 May 2021

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

4. Let n be a positive integer. The islands of the MO-Archipelago are arranged in a regular equilateral triangular unitary grid to form a big equilateral triangle of side length n. The beloved Governor Henning is in charge of building bridges between every pair of islands that are a distance of 1 apart. For every island i, Henning chooses two real numbers x_i and y_i that satisfy $x_i^2 + y_i^2 = 1$. The cost of a bridge between islands i and j is then given by $1 + x_i x_j + y_i y_j$. Determine the minimal amount of money needed to build all the bridges.

Remark: Below is a map of the MO-Archipelago in the case $n = 4$.



5. Let n be a positive integer. Some of the squares of a $3n \times 3n$ board are marked. For any marked square T, we denote by $\ell(T)$ the number of marked squares in the same row to the left of T and by $d(T)$ the number of marked squares in the same column below T. Determine the maximal number of marked squares given that $\ell(T) + d(T)$ is even for every marked square T.

6. We call a positive integer *silly* if the sum of its positive divisors is a square. Prove that there are infinitely many silly numbers.

Good Luck!

IMO-Selection 2021

Third exam

22 May 2021

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

7. Let n be a positive integer. Call a sequence of positive integers $a_1, a_2, \ldots, a_n$ *tame* if it satisfies

$$1 \cdot a_1 \leq 2 \cdot a_2 \leq \ldots \leq n \cdot a_n.$$

Determine the number of tame permutations of $1, 2, \ldots, n$.

8. Let ABC be a triangle such that $BC = CA$. Let D be a point inside the segment AB such that $AD < DB$. Let P and Q be two points inside the segments BC and CA respectively such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ intersect the segment CQ at E. The circumcircles of ABC and PQC intersect at C and F. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

9. Find all polynomials P with real coefficients having no repeated roots, such that for any complex number z, the equation $zP(z) = 1$ holds if and only if $P(z-1)P(z+1) = 0$.

Good Luck!

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Fourth exam

23 May 2021

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10. Prove that there are infinitely many positive integers n such that

$$n^2 + 1 \mid n!$$

holds.

11. Find all even functions $g\colon \mathbb{R} \to \mathbb{R}$ for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$

$$g(f(x) + y) = g(x) + g(y) + yf(x + f(x)).$$

Remark: An even function g is a function with the property $g(x) = g(-x)$ for every $x \in \mathbb{R}$.

12. Let ABC be an acute triangle, and I its incenter. Let A_1 be the intersection of AI and BC, and C_1 the intersection of CI and AB. Furthermore, let M and N be the midpoints of AI and CI, respectively. Inside the triangles AC_1I and A_1CI we choose points K and L such that $\angle AKI = \angle CLI = \angle AIC$, $\angle AKM = \angle ICA$ and $\angle CLN = \angle IAC$. Prove that the radii of the circumcircles of the triangles KIL and ABC are equal.

Good Luck!