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Second Round 2023

Zurich, Lausanne, Lugano
17 December 2022

Duration: 3 hours
Difficulty: The problems within one subject are ordered by difficulty.
Points: Each problem is worth 7 points.

Geometry

G1) Let ABC be a triangle satisfying $2 \cdot \angle CBA = 3 \cdot \angle ACB$. Let D and E be points on the side AC, such that BD and BE divide $\angle CBA$ into three equal angles and such that D lies between A and E. Furthermore, let F be the intersection of AB and the angle bisector of $\angle ACB$. Show that BE and DF are parallel.

G2) Let ω_1 be a circle with diameter JK. Let t be the tangent to ω_1 at J and let $U \neq J$ be another point on t. Let ω_2 be the smaller circle centred at U that touches ω_1 at one single point Y. Let I be the second intersection of JK with the circumcircle of triangle JYU and let F be the second intersection of KY with ω_2. Show that $FUJI$ is a rectangle.

Combinatorics

C1) During the World Cup, there are n different Panini stickers to collect. Marco's friends are trying to complete their collection, but nobody has a full set of stickers yet! A pair of his friends are said to be *wholesome* if their combined collection has at least one of each sticker. Marco knows the contents of everyone's collections, and wants to take them all to a restaurant for his birthday. However, he doesn't want any wholesome pairs sitting at the same table.

(i) Show that Marco might need to reserve at least n different tables.

(ii) Show that n tables will always be enough for Marco to achieve his goal.

C2) Let n be a positive integer. Roger has a $(2n+1) \times (2n+1)$ square garden. He puts down fences to divide his garden into rectangular plots. He wants to end up with exactly two horizontal $k \times 1$ plots and exactly two vertical $1 \times k$ plots for each **even** integer k between 1 and $2n+1$, as well as a single 1×1 square plot. How many different ways are there for Roger to do this?

Number Theory

N1) Determine all integer values that the expression

$$\frac{pq + p^p + q^q}{p+q}$$

can take, where p and q are both prime numbers.

N2) Determine all triples (a, b, p) of positive integers where p is prime and the equation

$$(a+b)^p = p^a + p^b$$

is satisfied.

Good luck!