Second round 2022

Zürich, Lausanne, Lugano

18 December 2021

Duration: 3 hours
Difficulty: The problems within one subject are ordered by difficulty.
Points: Each problem is worth 7 points.

Geometry

G1) Let k be a circle centred at O and let X, A, Y be three points on k in this order such that the tangent to the circumcircle of triangle OXA through X and the tangent to the circumcircle of OAY through Y are parallel. Show that $\angle XAY = 120°$ if A lies on the minor arc XY.

Remark: The minor arc XY is the shorter arc of the circle k linking X and Y.

G2) Let k_1 be a circle centred at M and ℓ a line tangent to k_1 at A. Let k_2 be a circle inside k_1 also tangent to ℓ at A. Let P be a point on ℓ different from A. The second tangent to k_1 through P touches k_1 at T. Let B be the second intersection of AT and k_2, and let C be the second intersection of PB and k_2. Show that $ATCM$ is a cyclic quadrilateral.

Combinatorics

C1) A school class of $n \geq 2$ children is taking several group pictures. For every group with at least one child, there is exactly one picture depicting this specific group. The pictures are now hung up in different rooms in the school, such that every child appears in at most one photo per room.

(i) Show that this is possible if the school has 2^{n-1} rooms.

(ii) Show that this is not possible if the school has less than 2^{n-1} rooms.

C2) There are 924 fans of the Liechtenstein football team from either Liechtenstein or Switzerland who have gathered to get the autographs of their favourite players. There are 11 players on the team, and every fan has exactly 6 favourite players. No two people from a given country share the same group of favourites, and in the end everyone got exactly one autograph from one of their favourite players. Show that there is a player who gave an autograph to both a Swiss and a Liechtensteiner person.

Number Theory

N1) Determine all pairs (m, p) of a positive integer m and a prime number p satisfying the equation

$$p^2 + pm = m^3.$$

N2) Let n be a positive integer and d a positive divisor of n. Show that if

$$\frac{d^2 + d + 1}{n + 1}$$

is an integer, then it is equal to 1.

Good luck!