Second round 2021

Zoom

19 December 2020

Duration: 3 hours
Difficulty: The problems within one subject are orderd by difficulty.
Points: Each problem is worth 7 points.

Geometry

G1) Let O be the centre of the circumcircle of an acute triangle ABC. The line AC intersects the circumcircle of the triangle ABO a second time at S. Prove that the line OS is perpendicular to the line BC.

G2) Let ABC be an acute triangle with $BC > AC$. The perpendicular bisector of the segment AB intersects the line BC at X and the line AC at Y. Let P be the projection of X on AC and let Q be the projection of Y on BC. Prove that the line PQ intersects the segment AB at its midpoint.

Remark: P being the projection of X on AC means that P lies on the line AC and PX is perpendicular to AC.

Combinatorics

C1) Anaëlle has $2n$ stones labelled $1, 2, 3, \ldots, 2n$ as well as a red box and a blue box. She wants to put each of the $2n$ stones into one of the two boxes such that the stones k and $2k$ are in different boxes for all $k = 1, 2, \ldots, n$. How many possibilities does Anaëlle have to do so?

Remark: Partial points are awarded for computing the number of possibilities for any particular integer $n \geq 3$.

C2) Let $n \geq 4$ and $k, d \geq 2$ be integers such that $k \cdot d \leq n$. The n contestants of the Mathematical Olympiad are sitting around a round table, waiting for Patrick to arrive. When Patrick arrives, he is unhappy about the situation because it violates the rules of social distancing. He therefore chooses k of the n contestants to stay and tells the others to leave the room such that between any two of the remaining k contestants, there are at least $d-1$ empty chairs. How many possibilities does Patrick have to do so if every chair was occupied in the beginning?

Number Theory

N1) Prove that for every integer $n \geq 3$ there exist positive integers $a_1 < a_2 < \ldots < a_n$ such that

$$a_k \mid (a_1 + a_2 + \ldots + a_n)$$

holds for every $k = 1, 2, \ldots, n$.

N2) Find all positive integers $n \geq 2$ such that, for every divisor $d > 1$ of n, we have

$$d^2 + n \mid n^2 + d.$$

Good luck!