SMO - Preliminary round 2018

Lausanne, Lugano, Zürich - 13 January 2018

Duration: 3 hours
Difficulty: The problems within one subject are orderd by difficulty.
Points: Each problem is worth 7 points.

Geometry

G1) Let ABC be a triangle and let $\gamma = \angle ACB$. Assume that $\gamma/2 < \angle BAC$ and $\gamma/2 < \angle CBA$ hold. Let D be the point on the side BC such that $\angle BAD = \gamma/2$. Let E be the point on the side CA such that $\angle EBA = \gamma/2$. Let furthermore F denote the intersection point of the angle bisector of $\angle ACB$ with the side AB. Prove that $EF + FD = AB$.

G2) Let $ABCD$ be an inscribed quadrilateral with circumcenter O such that the diagonals AC and BD are perpendicular. Let g be the line symmetric of the diagonal AC with respect to the angle bisector of $\angle BAD$. Prove that O lies on the line g.

Combinatorics

C1) SMO-Land has 1111 inhabitants. The eleven players of the national football team of Liechtenstein are distributing autographs to the inhabitants such that everybody gets at most one autograph from each player (i.e. every inhabitant gets from each player either one or zero autograph).

(a) How many distinct sets of autographs can be distributed by the players to some given inhabitant?

(b) After the distribution, it is noted that no two inhabitants received autographs from the exact same players. Prove that there are two inhabitants who, after putting together all their autographs, have exactly one autograph of each player.

C2) A building has 7 lifts and each lift only stops at 6 floors. However, for any two floors there is always a lift connecting them directly.

Prove that this building has at most 14 floors and that one can build such a building with 14 floors.

Number Theory

N1) Let $n \geq 2$ be a positive integer. Let $d_1, \ldots, d_r$ be all the positive divisors of n that are smaller that n. Determine every n for which

$$\operatorname{lcm}(d_1, \ldots, d_r) \neq n.$$

Remark: for $n = 18$ *we have for instance* $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 6, d_5 = 9$ *and thus* $\operatorname{lcm}(1, 2, 3, 6, 9) = 18$.

N2) Let m and n be positive integers and p a prime number for which $m < n < p$. Assume furthermore

$$p \mid m^2 + 1 \quad \text{and} \quad p \mid n^2 + 1.$$

Prove that

$$p \mid mn - 1.$$