SMO - Preliminary round 2017

Lausanne, Lugano, Zürich - 14 January 2017

Duration: 3 hours
Difficulty: The problems within one subject are orderd by difficulty.
Points: Each problem is worth 7 points.

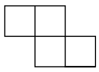
Geometry

G1) Let ABC be a triangle with $AB \neq AC$ and circumcircle k. The tangent of k at A intersects BC at P. The angle bisector of $\angle APB$ intersects AB at D and AC at E. Show that the triangle ADE is isosceles.

G2) Let ABC be a right-angled triangle with hypotenuse AB. A circle with center C intersects the segment AB twice at the points P and Q such that P lies between A and Q. Let R be the point on the segment BC with $\angle RAC = \frac{1}{2}\angle PCQ$ and let S be the point on the segment AC with $\angle CBS = \frac{1}{2}\angle PCQ$. Further let T be the intersection of the lines CP and AR, and U be the intersection of the lines CQ and BS. Show that $RSTU$ is a cyclic quadrilateral.

Combinatorics

K1) What is the maximal number of skew-tetrominos that can be placed on a 8×9 board without overlapping?



Remark: Tetrominos may be rotated and mirrored.

K2) Let $m, n \geq 2$ be positive integers. We have four colours and want to colour each unit square of a $m \times n$ board with one of them such that in every 2×2 square all four colours occur. How many different possibilities are there?

Remark: We count two possibilities as different if there is at least one square which received different colours.

Number Theory

Z1) Determine all pairs (m, n) of positive integers such that

$$\operatorname{lcm}(m, n) - \gcd(m, n) = \frac{mn}{5}.$$

Z2) Let a and b be positive integers such that

$$\frac{3a^2 + b}{3ab + a}$$

is an integer. Which values can be taken on by this expression?