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Final Round 2025

First Exam

Aarburg
March 21, 2025

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

1. Let $ABCD$ be a cyclic quadrilateral without parallel sides. Let points X and Y lie on DA such that $BX \parallel CD$ and $CY \parallel AB$. Let Z be the intersection of BX and CY, and M be the midpoint of the segment BC.

Prove that MZ is perpendicular to the line joining the circumcenters of triangles ABX and CDY.

2. Determine all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$y \cdot \min\Big(f(xy),\ f(x)\Big) = \min\Big(f\Big(\frac{x}{y}\Big),\ f(x)\Big)$$

for all $x, y \in \mathbb{R}_{>0}$.

3. Let n be a positive integer. A group of n penguins swim in n races, in each of which they are ranked from 1st to nth position with no draws. A penguin is eligible for a rating (a, b) for positive integers a and b, if in at least a races they finished in any of the first b positions. Their final score is the maximum possible value $a - b$ across all ratings for which they are eligible.

Find the maximum possible sum of all scores of the n penguins.

4. Determine all infinite sequences $a_1, a_2, \ldots$ of positive integers such that, for any integer $n \geq 2$, both the arithmetic and geometric mean of any n consecutive terms of the sequence are integers.

Remark: Integers $x_1, \ldots, x_k > 0$ have arithmetic mean $\frac{x_1+\cdots+x_k}{k}$ and geometric mean $\sqrt[k]{x_1 \cdots x_k}$.

Good luck!

Final Round 2025

Second Exam

Aarburg

March 22, 2025

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

5. Determine all triples of positive integers (p, q, a) such that p and q are prime numbers and

$$p^q - q^a = 2025.$$

6. Let n, a, b be positive integers with $n \geq 2$. Aru and Wero are playing a game on a $n \times n$ grid. At the beginning, there is a single pebble in the bottom-left corner. A move consists of removing one pebble from some non-empty square S and performing at least one (possibly both) of:

- adding a pebbles to the right neighbour of S;
- adding b pebbles to the upper neighbour of S.

Aru starts and they alternate moves. The first player that cannot make a move loses. Determine who, if anyone, has a winning strategy in terms of n, a, b.

7. Determine all sequences $x_1, x_2, \ldots, x_n$ of rational numbers, such that for all natural numbers m,

$$\frac{(x_1)^m + (x_2)^m + \cdots + (x_n)^m}{n}$$

is equal to some rational number raised to the power of m.

8. Let ABC be an acute triangle with $AB = AC$. Let D and E be points on the segments AB and AC, respectively. Let ω_1 be the circle with center D and radius DB and let ω_2 be the circle with center E and radius EC. Assume that ω_1 and ω_2 intersect twice and let P be the intersection closest to BC. Denote by $F \neq B$ and $G \neq C$ the intersections of BC with ω_1 and ω_2, respectively. Finally, let DF and EG intersect in Q, and let the angle bisectors of QDP and PEQ intersect in S.

Show that the circumcircles of SDQ and SEP are tangent.

Good luck!