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Final Round 2024

First Exam

Zürich
March 2, 2024

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

1. If a and b are positive integers, we say that a *almost divides* b if a divides at least one of $b-1$ and $b+1$. We call a positive integer n *almost prime* if the following holds: for any positive integers a, b such that n almost divides ab, we have that n almost divides at least one of a and b. Determine all almost prime numbers.

2. Let ABC be a triangle with incenter I, and let J be the reflection of I with respect to line BC. Let K be the second intersection of line BC with the circumcircle of triangle CIJ, and L be the second intersection of line BI with the circumcircle of triangle AIK. Prove that the lines BC and JL are parallel.

3. Suppose that a, b, c, d are positive real numbers satisfying $ab^2 + ac^2 \geq 5bcd$. Determine the smallest possible value of

$$(a^2 + b^2 + c^2 + d^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right).$$

4. Determine the maximal length L of a sequence $a_1, \ldots, a_L$ of positive integers satisfying both of the following properties:

- every term in the sequence is less than or equal to 2^{2024}, and
- there does not exist a consecutive subsequence $a_i, a_{i+1}, \ldots, a_j$ (where $1 \leq i \leq j \leq L$) with a choice of signs $s_i, s_{i+1}, \ldots, s_j \in \{1, -1\}$ for which

$$s_i a_i + s_{i+1} a_{i+1} + \cdots + s_j a_j = 0.$$

Good luck!

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Second Exam

Zürich
March 3, 2024

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

5. The icy ballroom of the White Witch is shaped like a square, and the floor is covered by $n \times n$ identical square tiles. Additionally, between some pairs of adjacent tiles there are magic edges. The White Witch's loyal servant Edmund is tasked with cleaning the ballroom. He starts in one of the corner tiles and may move up, down, left and right. However, as the floor is slippery, he will slide in that direction until he hits a wall or a magic edge. On the upside, he is wearing special shoes that clean each tile he passes over.

Determine the minimum number of magic edges that need to be placed for Edmund to be able to clean all the tiles.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y)f(x-y) \geq f(x)^2 - f(y)^2$$

for every $x, y \in \mathbb{R}$. Assume that the inequality is strict for some $x_0, y_0 \in \mathbb{R}$.

Prove that $f(x) \geq 0$ for every $x \in \mathbb{R}$ or $f(x) \leq 0$ for every $x \in \mathbb{R}$.

7. Determine all positive integers n satisfying all of the following properties:

- there exist exactly three distinct prime numbers dividing n,
- n is equal to $\binom{m}{3}$ for some positive integer m, and
- $n+1$ is a perfect square.

8. Let $ABCD$ be a cyclic quadrilateral with $\angle BAD < \angle ADC$. Let M be the midpoint of the arc CD not containing A. Suppose there is a point P inside $ABCD$ such that $\angle ADB = \angle CPD$ and $\angle ADP = \angle PCB$. Prove that the lines AD, PM, BC are concurrent.

Good luck!