MATHEMATICAL.
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OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Final Round 2023

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Aarburg
March 10, 2023
First Exam

1. Let ABC be an acute triangle with incentre I. On its circumcircle, let M_A, M_B and M_C be the midpoints of minor arcs BC, CA and AB respectively. Prove that the reflection of M_A over the line IM_B lies on the circumcircle of the triangle IM_BM_C.

2. The wizards Albus and Brian are playing a game on a square of side length $2n+1$ metres surrounded by lava. In the centre of the square there sits a toad. In a turn, a wizard chooses a direction parallel to a side of the square and enchants the toad. This will cause the toad to jump d metres in the chosen direction, where d is initially equal to 1 and increases by 1 after each jump. The wizard who sends the toad into the lava loses. Albus begins and they take turns. Depending on n, determine which wizard has a winning strategy.

3. Let x, y and $a_0, a_1, a_2, \ldots$ be integers satisfying $a_0 = a_1 = 0$ and

$$a_{n+2} = x \cdot a_{n+1} + y \cdot a_n + 1$$

for all integers $n \geq 0$. Let p be any prime number. Show that $\gcd(a_p, a_{p+1})$ is either equal to 1 or greater than $\sqrt{p}$.

4. Determine the smallest possible value of the expression

$$\frac{ab+1}{a+b} + \frac{bc+1}{b+c} + \frac{ca+1}{c+a},$$

where $a, b, c \in \mathbb{R}$ satisfy $a+b+c = -1$ and $abc \leq -3$.

Good luck!

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Aarburg
March 11, 2023
Second Exam

5. Let D be the set of real numbers excluding -1. Find all functions $f\colon D \to D$ such that for all $x, y \in D$ satisfying $x \neq 0$ and $y \neq -x$, the equality

$$\Big(f(f(x)) + y\Big) f\left(\frac{y}{x}\right) + f(f(y)) = x$$

holds.

6. Determine all integers $n \geq 3$ such that

$$n! \mid \prod_{\substack{p<q\leq n \\ p,q \text{ prime}}} (p+q).$$

Remark: The expression on the right-hand side denotes the product over all sums of two distinct primes less than or equal to n. For $n = 6$, this is equal to $(2+3)(2+5)(3+5)$.

7. In the acute triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E respectively. Points $X \neq A$ and $Y \neq A$ lie on the circumcircles of triangles ABD and ACE respectively, such that $DA = DX$ and $EA = EY$. Prove that $BCXY$ is a cyclic quadrilateral.

8. Let n be a positive integer. Kimiko starts with n piles of pebbles each containing a single pebble. She can take an equal number of pebbles from two existing piles and combine the removed pebbles to create a new pile. Determine, in terms of n, the smallest number of nonempty piles Kimiko can end up with.

Good luck!