MATHEMATICAL. OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Final round 2022

First exam

18 March 2022

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

1. Let k be a circle with centre M and let AB be a diameter of k. Furthermore, let C be a point on k such that $AC = AM$. Let D be the point on the line AC such that $CD = AB$ and C lies between A and D. Let E be the second intersection of the circumcircle of BCD with line AB and F be the intersection of the lines ED and BC. The line AF cuts the segment BD in X. Determine the ratio BX/XD.

2. Let n be a positive integer. Prove that the numbers

$$1^1,\ 3^3,\ 5^5,\ \ldots,\ (2^n-1)^{2^n-1}$$

all give different remainders when divided by 2^n.

3. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that both

- $f(f(m)f(n)) = mn$
- $f(2022a+1) = 2022a+1$

hold for all positive integers m, n and a.

4. Let $n \geq 2$ be an integer. Switzerland and Liechtenstein are performing their annual festive show. There is a field subdivided into $n \times n$ squares, in which the bottom-left square contains a red house with k Swiss gymnasts, and the top-right square contains a blue house with k Liechtensteiner gymnasts. Every other square only has enough space for a single gymnast at a time. Each second either a Swiss gymnast or a Liechtensteiner gymnast moves. The Swiss gymnasts moves to either the square immediately above or to the right and the Liechtensteiner gymnasts moves either to the square immediately below or to the left. The goal is to move all the Swiss gymnasts to the blue house and all the Liechtensteiner gymnasts to the red house, with the caveat that a gymnast cannot enter a house until all the gymnasts of the other nationality have left. Determine the largest k in terms of n for which this is possible.

Good luck!

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Second exam
19 March 2022

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5. For an integer $a \geq 2$, denote by $\delta(a)$ the second largest divisor of a. Let $(a_n)_{n\geq 1}$ be a sequence of integers such that $a_1 \geq 2$ and

$$a_{n+1} = a_n + \delta(a_n)$$

for all $n \geq 1$. Prove that there exists a positive integer k such that a_k is divisible by 3^{2022}.

6. Let $n \geq 3$ be an integer. Annalena has infinitely many cowbells in each of n different colours. Given an integer $m \geq n+1$ and a group of m cows standing in a circle, she is tasked with tying one cowbell around the neck of every cow so that every group of $n+1$ consecutive cows have cowbells of all the possible n colours. Prove that there are only finitely many values of m for which this is not possible and determine the largest such m in terms of n.

7. Let $n > 6$ be a perfect number. Let $p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ be the prime factorisation of n where we assume that $p_1 < p_2 < \ldots < p_k$ and $a_i > 0$ for all $i = 1, \ldots, k$. Prove that a_1 is even.

Remark: An integer $n \geq 2$ is called a perfect number *if the sum of its positive divisors, excluding n itself, is equal to n. For example, 6 is perfect, as its positive divisors are $\{1, 2, 3, 6\}$ and $1 + 2 + 3 = 6$.*

8. Let ABC be a triangle and let P be a point in the interior of the side BC. Let I_1 and I_2 be the incenters of the triangles APB and APC, respectively. Let X be the closest point to A on the line AP such that XI_1 is perpendicular to XI_2. Prove that the distance AX is independent of the choice of P.

Good luck!