MATHEMATICAL.
OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Final round 2021

First exam

19 February, 2021

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

1. Let (m, n) be a pair of positive integers. Julia has carefully planted m rows of n dandelions in an $m \times n$ array in her back garden. Now, Jana and Viviane decide to play a game with a lawnmower they just found. Taking alternating turns and starting with Jana, they can mow down all the dandelions in a straight horizontal or vertical line (and they must mow down at least one dandelion!). The winner is the player who mows down the final dandelion. Determine all pairs (m, n) for which Jana has a winning strategy.

2. Let ABC be an acute triangle with $AB = AC$ and let D be a point on the side BC. The circle with centre D passing through C intersects the circumcircle of ABD in P and Q, where Q is the point closer to B. The line BQ intersects AD in X and AC in Y. Prove that $PDXY$ is cyclic.

3. Find all finite sets S of positive integers with at least two elements, such that if $m > n$ are two elements of S, then

$$\frac{n^2}{m-n}$$

is also an element of S.

4. The real numbers a, b, c, d are positive and satisfy $(a+c)(b+d) = ac + bd$. Find the minimum of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Good Luck!

MATHEMATICAL.
OLYMPIAD.CH
MATHEMATIK-OLYMPIADE
OLYMPIADES DE MATHÉMATIQUES
OLIMPIADI DELLA MATEMATICA

Final round 2021

Second exam

20 February, 2021

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

5. For which integers $n \geq 2$ can we arrange the numbers $1, 2, \ldots, n$ in a row, such that for all integers $1 \leq k \leq n$ the sum of the first k numbers in the row is divisible by k?

6. Let $\mathbb{N}$ be the set of positive integers. Let $f \colon \mathbb{N} \to \mathbb{N}$ be a function such that for every $n \in \mathbb{N}$

$$f(n) - n < 2021 \quad \text{and} \quad \underbrace{f(f(\cdots f(f}_{f(n)}(n))\cdots)) = n.$$

Prove that $f(n) = n$ for infinitely many $n \in \mathbb{N}$.

7. Let $m \geq n$ be positive integers. Frieder is given mn posters of Linus with different integer dimensions $k \times l$ with $1 \leq k \leq m$ and $1 \leq l \leq n$. He must put them all up one by one on his bedroom wall without rotating them. Every time he puts up a poster, he can either put it on an empty spot on the wall, or on a spot where it entirely covers a single visible poster and does not overlap any other visible poster. Determine the minimal area of the wall that will be covered by posters.

8. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 20°$. Let D be the point on the side AB such that $\angle BCD = 70°$. Let E be the point on the side AC such that $\angle CBE = 60°$. Determine the value of the angle $\angle CDE$.

Good Luck!