SMO - Final round 2018

First exam - 16 January 2018

Duration: 4 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. The cells of an 8×8 chessboard are all coloured in white. A move consists in inverting the colours of a rectangle 1×3 horizontal or vertical (the white cells become black and conversely). Is it possible to colour all the cells of the chessboard in black in a finite number of moves ?

2. Let a, b and c be natural numbers. Determine the smallest value that the following expression can take:

$$\frac{a}{\gcd(a+b, a-c)} + \frac{b}{\gcd(b+c, b-a)} + \frac{c}{\gcd(c+a, c-b)}.$$

Remark: $\gcd(6,0) = 6$ *and* $\gcd(3,-6) = 3$.

3. Determine all natural integers n for which there is no triplet (a, b, c) of natural numbers such that:

$$n = \frac{a \cdot \operatorname{lcm}(b,c) + b \cdot \operatorname{lcm}(c,a) + c \cdot \operatorname{lcm}(a,b)}{\operatorname{lcm}(a,b,c)}.$$

4. Let D be a point inside an acute triangle ABC such that $\angle BAD = \angle DBC$ and $\angle DAC = \angle BCD$. Let P be a point on the circumcircle of triangle ADB. We assume that P lies outside of the triangle ABC. A line through P intersects the ray BA in X and the ray CA in Y such that $\angle XPB = \angle PDB$. Prove that BY and CX intersect on AD.

Remark: For two points F and G, the ray FG is the set of points on the line FG that lies on the same side of F than G.

5. Prove that there exists no function $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for every $x, y \in \mathbb{R}_{>0}$

$$f(xf(x) + yf(y)) = xy.$$

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6. Let k be the incircle of triangle ABC with center I. The circle k touches the sides BC,CA and AB at the points D, E and F respectively. Let G be the intersection point of the segment AI with the circle k. We assume that the lines BE and FG are parallel. Prove that $BD = EF$.

7. Let n be a natural integer and let k be the number of ways to write n as the sum of one or more consecutive natural integers. Prove that k is equal to the number of odd positive divisors of n.

Example: 9 has three positive odd divisors and $9 = 9,\ 9 = 4 + 5,\ 9 = 2 + 3 + 4$.

8. Let a, b, c, d and e be positive real numbers. Determine the largest value that the following expression can take:

$$\frac{ab + bc + cd + de}{2a^2 + b^2 + 2c^2 + d^2 + 2e^2}.$$

9. Let n be a natural integer and G be the set of points (x, y) in the plane such that x and y are integers with $1 \leq x, y \leq n$. A subset of G is called *parallelogramfree* if it does not contain four non-collinear points that are the vertices of a parallelogram. How many points at most can a parallelogramfree subset contain ?

10. Let $p \geq 2$ be a prime number. Arnaud and Louis alternatively choose an index $i \in \{0, 1, \ldots, p-1\}$ that has not already been chosen and a digit $a_i \in \{0, 1, \ldots, 9\}$. Arnaud starts. Once every index has been chosen, they compute the following sum:

$$a_0 + a_1 \cdot 10 + \ldots + a_{p-1} \cdot 10^{p-1} = \sum_{i=0}^{p-1} a_i \cdot 10^i.$$

If the sum is divisible by p, Arnaud wins. Otherwise Louis wins. Prove that Arnaud has a winning strategy.