SMO Final Round 2006

first exam - 31 march 2006

Time: 4 hours
Every problem is worth 7 points.

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ we have

$$yf(2x) - xf(2y) = 8xy(x^2 - y^2).$$

2. Let ABC be an equilateral triangle and let D be a point in the interior of the side BC. A circle touches BC in D and intersects the sides AB and AC in the interior points M, N and P, Q, respectively. Prove that

$$|BD| + |AM| + |AN| = |CD| + |AP| + |AQ|.$$

3. Compute the sum of the digits of the following number

$$9 \times 99 \times 9999 \times \cdots \times \underbrace{99\ldots99}_{2^n},$$

where the number of nines doubles in each factor.

4. A circle with circumference $6n$ is divided by $3n$ points into n intervalls of lengths 1, 2 and 3 respectively. Show that there can always be found two of these points that lie diametrically opposite.

5. A circle k_1 is contained in another circle k_2 touching it in the point A. A line passing through A intersects k_1 again in B and k_2 in C. The tangent to k_1 at B intersects k_2 in the points D and E. The tangents to k_1 through C touch k_1 in the points F and G. Prove that D, E, F and G lie on a circle.

Good luck!

SMO Final Round 2006

second exam - 1 april 2006

Time: 4 hours
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6. Three or more players participated in a tennis tournament. Every two players played exactly once against each other end every player won at least one of his matches. Show that there are three players A, B, C such that A won against B, B against C and C against A.

7. Let $ABCD$ be a cyclic quadrilateral with $\angle ABC = 60°$. Suppose $|BC| = |CD|$. Prove that

$$|CD| + |DA| = |AB|.$$

8. People from n different countries sit at a round table. Assume that for every two members of the same country their neighbours sitting next to them on the right hand side are from different countries. Find the largest possible number of people sitting around the table.

9. Let a, b, c, d be real numbers. Prove the following inequality.

$$(a^2 + b^2 + 1)(c^2 + d^2 + 1) \geq 2(a + c)(b + d).$$

10. Decide wether there exists an integer $n > 1$ with the following properties:

(a) n is not a prime number

(b) $a^n - a$ is divisible by n for all intergers a.